DEMONSTRATIO MATHEMATICA
Vol. X No 3-4 1977

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INTEGRABILITY CONDITIONS OF A F (K, K-2)-STRUCTURE SATISFYING $F^K+F^{K-2}=0$

Introduction

Yano, Houh and Chen [1] have studied the structures defined by a tensor field ϕ of the type (1,1) satisfying $\phi^4 \pm \phi^2 = 0$. Gadea and Cordero [2] have obtained the integrability conditions of these structures. We shall obtain in this paper the integrability conditions of a generalised F(K,K-2)-structure satisfying $F^K + F^{K-2} = 0$, where F is a non-null tensor field of the type (1,1). Besides this we have also obtained the conditions of partial integrability (by introducing s_K-partial integrability and t_K-partial integrability) and the integrability of the generalised F(K-K-2)-structure in terms of its Nijenhuis tensor for K even.

1. The operators s and t: Let M_n be an n-dimensional differentiable manifold of class C^∞ equipped with a (1,1) tensor field $F(F \neq 0)$ and of class C^∞ satisfying

$$(1.1)\qquad \begin{cases} n = 2m, \\ F^K + F^{K-2} = 0, \\ (2 \text{ rank } F - \text{rank } F^{K-2}) = \dim M_n, \end{cases}$$

where K is even.

The operators s and t have been defined as follows

$$(1.2) \qquad s = (-1)^{\frac{K}{2}-1} F^{K-2}, \quad t = I - (-1)^{\frac{K}{2}-1} F^{K-2},$$

I denoting the identity operators. Then we have

T h e o r e m 1.1. For a tensor field $F (F \neq 0)$ satisfying (1.1), the operators s and t defined by (1.2) and applied to the tangent space at a point of the manifold are complementary projection operators.

P r o o f . In consequence of (1.1) and (1.2), we have

$$(1.3) \qquad s + t = I,$$

$$(1.4) \quad \begin{cases} s^2 = (-1)^{K-2} F^{2K-4} = F^K F^{K-4} = - F^{K-2} F^{K-(2+2)} = \\ = - F^K F^{K-6} = (-1)^2 F^{K-2} F^{K-(2+4)} = \\ = (-1)^2 F^K F^{K-8} = \ldots = (-1)^{\frac{1}{2}(K-2)} F^{K-2} F^{K-(2+K-2)} = \\ = (-1)^{\frac{K}{2}-1} F^{K-2} = s, \end{cases}$$

$$(1.5) \quad \begin{cases} t^2 = I + (-1)^{K-2} F^{2K-4} - 2 (-1)^{\frac{K}{2}-1} F^{K-2} = \\ = I - (-1)^{\frac{K}{2}-1} F^{K-2} = t, \end{cases}$$

$$(1.6) \qquad st = ts = (-1)^{\frac{K}{2}-1} F^{K-2} - (-1)^{K-2} F^{2K-4} = 0.$$

This proves the theorem.

Let S and T be the complementary distributions corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r, then from (1.1) we get

$$\dim S = (2r-n) \quad \text{and} \quad \dim T = (2n-2r).$$

Here dimensions of S and T are both even. Obviously $n \leq 2r \leq 2n$. Such a structure has been called a generalised F(K,K-2)-structure of rank r and the manifold M_n with this structure a F(K,K-2)-manifold.

T h e o r e m 1.2. For a tensor field $F (F \neq 0)$ satisfying (1.1) and the operators s and t defined by (1.2), we have

$$F^{K-2}s = sF^{K-2} = F^{K-2}, \quad F^{K-2}t = tF^{K-2} = 0 \tag{1.7}$$

and

$$F^{K-1}s = F^{K-1}, \quad F^{K-1}t = 0. \tag{1.8}$$

P r o o f . The proof of the theorem follows by virtue of the equations (1.2) and (1.4).

T h e o r e m 1.3. For a tensor field $F (F \neq 0)$ satisfying (1.1) and the operators s and t defined by (1.2), we have

$$Fs = sF = (-1)^{\frac{K}{2}-1} F^{K-1}, \quad Ft=tF=F-(-1)^{\frac{K}{2}-1}F^{K-1} \tag{1.9}$$

and

$$F^2s = -s, \quad F^2t = F^2 + (-1)^{\frac{K}{2}-1} F^{K-2}. \tag{1.10}$$

The proof is obvious.

T h e o r e m 1.4. F(K,K-2)-structure of maximal rank is an almost complex structure.

P r o o f . If the rank of F is maximal, $r = n$. Then $t = 0$. Thus F satisfies

$$I - (-1)^{\frac{K}{2}-1} F^{K-2} = 0. \tag{1.11}$$

Applying F twice to (1.11) and using (1.1), we get

$$F^2 + (-1)^{\frac{K}{2}-1} F^{K-2} = 0,$$

which in view of (1.11) yields

$$F^2 + I = 0.$$

This proves the theorem.

T h e o r e m 1.5. F(K,K-2)-structure of minimal rank is a F(K-2)-structure.

P r o o f . If the rank of F is minimal, 2r=n. Then s = 0. Thus F satisfies $F^{K-2} = 0$. Following the general nomenclature, we call such a structure a F(K-2)-structure.

2. Nijenhuis tensor of F(K-K-2)-structure

Let F be a F(K,K-2)-structure of rank r when K is even. Then the Nijenhuis tensor N(X,Y) of F is

$$(2.1) \quad N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y].$$

Therefore, in consequence of (1.9) and (2.1), we have

$$(2.2) \quad \begin{cases} N(sX,sY) = [(-1)^{\frac{K}{2}-1}F^{K-1}X,(-1)^{\frac{K}{2}-1}F^{K-1}Y] - \\ - F[(-1)^{\frac{K}{2}-1}F^{K-1}X,sY] - F[sX,(-1)^{\frac{K}{2}-1}F^{K-1}Y] + \\ + F^2[sX,sY]. \end{cases}$$

$$(2.3) \quad \begin{cases} N(sX,tY) = [(-1)^{\frac{K}{2}-1}F^{K-1}X,FY - (-1)^{\frac{K}{2}-1}F^{K-1}Y] - \\ - F[(-1)^{\frac{K}{2}-1}F^{K-1}X,tY] - F[sX,FY - (-1)^{\frac{K}{2}-1}F^{K-1}Y] + \\ + F^2[sX,tY], \end{cases}$$

$$(2.4)\begin{cases} N(tX,sY) = \left[FX - (-1)^{\frac{K}{2}-1}F^{K-1}X, (-1)^{\frac{K}{2}-1}F^{K-1}Y\right] - \\ - F\left[FX - (-1)^{\frac{K}{2}-1}F^{K-1}X, sY\right] - \\ - F\left[tX, (-1)^{\frac{K}{2}-1}F^{K-1}Y\right] + F^2\left[tX,sY\right], \end{cases}$$

$$(2.5)\begin{cases} N(tX,tY) = \left[FX - (-1)^{\frac{K}{2}-1}F^{K-1}X, FY - (-1)^{\frac{K}{2}-1}F^{K-1}Y\right] - \\ - F\left[FX - (-1)^{\frac{K}{2}-1}F^{K-1}X, tY\right] - \\ - F\left[tX, FY - (-1)^{\frac{K}{2}-1}F^{K-1}Y\right] + F^2\left[tX,tY\right]. \end{cases}$$

Equations (2.2), (2.3), (2.4) and (2.5), in consequence of (1.3) and (2.1) yield

$$(2.6)\quad N(X,Y) = N(sX,sY) + N(sX,tY) + N(tX,sY) + N(tX,tY).$$

If the distribution S is integrable, $N(sX,sY)$ is exactly the Nijenhuis tensor of $F|S \overset{\text{def}}{=\!=} F_S$. If the distribution T is integrable $N(tX,tY)$ is exactly the Nijenhuis tensor of $F|T \overset{\text{def}}{=\!=} F_T$.

Let $\mathcal{L}_Y F$ be Lie derivative of the tensor field F with respect to a vector field Y. Then we have

$$(2.7)\qquad (\mathcal{L}_Y F)X = F[X,Y] - [FX,Y],$$

where $\mathcal{L}_Y F$ is a tensor field of the same type as F.

Now in view of (2.1) and (2.7), we get

$$(2.8)\qquad N(sX,tY) = F(\mathcal{L}_{tY}F)sX - (\mathcal{L}_{FtY}F)sX$$

and

$$(2.9)\qquad N(tX,sY) = F(\mathcal{L}_{sY}F)tX - (\mathcal{L}_{FsY}F)tX.$$

3. Integrability conditions

In this section, we shall obtain the partial integrability conditions of the $F(K,K-2)$-structure, when K is even.

T h e o r e m 3.1. The following conditions hold:
(i) the distribution S is integrable iff $t\cdot N(sX,sY) = 0$;
(ii) the distribution T is integrable iff $s\cdot N(tX,tY) = 0$,
for any two vector fields X and Y.

P r o o f : We know that for any two vector fields X and Y, the distributions S and T are integrable if and only if $t[sX,sY] = 0$ and $s[tX,tY] = 0$ respectively. Thus in view of (1.6), (1.8), (1.9) and (2.1), the theorem follows.

T h e o r e m 3.2. The distributions S and T are both integrable if and only if

$$N(X,Y) = s\cdot N(sX,sY) + N(sX,tY) + N(tX,sY) + t\cdot N(tX,tY) \tag{3.1}$$

for any two vector fields X and Y.

P r o o f . In consequence of (1.3), equation (2.6) can be written as

$$\begin{aligned} N(X,Y) = {} & s\cdot N(sX,sY) + t\cdot N(sX,sY) + N(sX,tY) + \\ & + N(tX,sY) + s\cdot N(tX,tY) + t\cdot N(tX,tY). \end{aligned} \tag{3.2}$$

Now the result follows by virtue of the equation (3.2) and Theorem 3.1.

T h e o r e m 3.3. If the distribution S is integrable, a necessary and sufficient condition for the almost complex structure defined by $F|S = F_S$ on each integral manifold of S to be integrable is that, for any two vector fields X and Y

$$N(sX,sY) = 0, \tag{3.3}$$

which is equivalent to

$$s\cdot N(sX,sY) = 0. \tag{3.4}$$

P r o o f . Suppose that the distribution S is integrable, then F_S induces on each integral manifold of S an almost complex structure. The induced structure is integrable iff its Nijenhuis tensor vanishes identically. Thus the theorem follows.

D e f i n i t i o n 3.1. We say that the F(K,K-2)-structure is s_K- partially integrable if the distribution S is integrable and the almost complex structure F_S induced from F on each integral manifold of S is also integrable.

T h e o r e m 3.4. For any two vector fields X and Y, a necessary and sufficient condition for the F(K.K-2)-structure to be s_K-partially integrable is that

$$N(sX,sY) = 0. \tag{3.5}$$

The proof of the theorem follows from Theorems 3.1 (i) and 3.3.

T h e o r e m 3.5. If the distribution T is integrable, a necessary and sufficient condition for the F(K-2)-structure defined by $F|T = F_T$ on each integral manifold of T to be integrable is that, for any two vector fields X and Y

$$N(tX,tY) = 0, \tag{3.6}$$

which is equivalent to

$$t\cdot N(tX,tY) = 0. \tag{3.7}$$

The proof follows from the pattern of the proof of Theorem 3.3.

D e f i n i t i o n 3.2. We say that the F(K,K-2)-structure is t_K-partially integrable if the distribution T is integrable and the F(K-2)-structure F_T induced from F on each integral manifold of T is also integrable.

T h e o r e m 3.6. For any two vector fields X and Y, a necessary and sufficient condition for the $F(K,K-2)$-structure to be t_K-partially integrable is that

$$N(tX,tY) = 0. \tag{3.8}$$

The proof of the theorem follows from Theorems 3.1 (ii) and 3.5.

D e f i n i t i o n 3.3. We say that a $F(K,K-2)$-structure is partially integrable if and only if it is s_K-partially integrable and t_K-partially integrable simultaneously.

T h e o r e m 3.7. For any two vector fields X and Y, a necessary and sufficient condition for the $F(K,K-2)$-structure to be partially integrable is that

$$N(X,Y) = N(sX,tY) + N(tX,sY). \tag{3.9}$$

P r o o f . The proof of the theorem follows by virtue of the equations (2.6), (3.5) and (3.8).

4. Conditions $N(sX,tY) = 0$ and $N(tX,sY) = 0$

In this section, we shall obtain the integrability conditions of the $F(K,K-2)$-structure by means of the conditions $N(sX,tY) = 0$ and $N(tX,sY) = 0$, when K is even.

T h e o r e m 4.1. For any vector fields X and Y, the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically if and only if

$$N(sX,tY) = 0. \tag{4.1}$$

P r o o f . In view of (2.8), we have

$$N(sX,tY) = 0 \quad \text{if and only if} \quad F(\mathcal{L}_{tY}F)sX = (\mathcal{L}_{FtY}F)sX.$$

Thus, if $N(sX,tY) = 0$, we obtain

$$(-1)^{\frac{K}{2}-1}F^{K-2}(\mathcal{L}_{tY}F)sX = (-1)^{\frac{K}{2}-1}F^{K-3}(\mathcal{L}_{FtY}F)sX =$$

$$= (-1)^{\frac{K}{2}-1}F^{K-4}(\mathcal{L}_{F^2tY}F)sX = \dots = (-1)^{\frac{K}{2}-1}F^{K-K}(\mathcal{L}_{F^{K-2}tY}F)sX = 0$$

in consequence of (1.7). That is, in view of (1.2), the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y.

Theorem 4.2. For any vector fields X and Y, the tensor field $t(\mathcal{L}_{sY}F)t$ vanishes identically if and only if

$$N(tX,sY) = 0. \tag{4.2}$$

Proof. In view of (2.9), we have

$$N(tX,sY) = 0 \quad \text{if and only if} \quad F(\mathcal{L}_{sY}F)tX = (\mathcal{L}_{FsY}F)tX.$$

Thus, if $N(tX,sY) = 0$, we obtain

$$(-1)^{\frac{K}{2}-1} F^{K-2}(\mathcal{L}_{sY}F)tX = (-1)^{\frac{K}{2}-1} F^{K-3}(\mathcal{L}_{FsY}F)tX = \dots =$$

$$= (-1)^{\frac{K}{2}-1} F^{K-K}(\mathcal{L}_{F^{K-2}sY}F)tX = (-1)^{\frac{K}{2}-1} (\mathcal{L}_{F^{K-2}Y}F)tX = (\mathcal{L}_{sY}F)tX,$$

in consequence of (1.2) and (1.7). Hence

$$(I - (-1)^{\frac{K}{2}-1} F^{K-2})(\mathcal{L}_{sY}F)tX = 0.$$

That is, in view of (1.2), the tensor field $t(\mathcal{L}_{sY}F)t$ vanishes identically for any vector field Y.

When the distributions S and T are both integrable, we can choose a local coordinate system such that S's are represented by putting (2n-2r) local coordinates constant and T's by putting the other (2r-n) coordinates constant. We call such a coordinate system an "adapted coordinate system".

It can be supposed that in an adapted coordinate system, the projection operators s and t have the components of the form

(4.3) $$s = \begin{pmatrix} I_{2r-n} & 0 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 \\ 0 & I_{2n-2r} \end{pmatrix}$$

respectively; where I_{2r-n} is a unit matrix of order $(2r-n)$ and I_{2n-2r} is that of order $(2n-2r)$.

Since the distributions S and T are integrable, $FS \subset S$ and $FT \subset T$. Therefore the tensor F has the components of the form

(4.4) $$F = \begin{pmatrix} F_{2r-n} & 0 \\ 0 & F_{2n-2r} \end{pmatrix}$$

in an adapted coordinate system, where F_{2r-n} and F_{2n-2r} are square matrices of order $(2r-n) \times (2r-n)$ and $(2n-2r) \times (2n-2r)$ respectively.

Thus the Lie derivative $\mathcal{L}_{tY}F$ has components of the form

(4.5) $$\mathcal{L}_{tY}F = \begin{pmatrix} L' & 0 \\ 0 & L'' \end{pmatrix}$$

for any vector field tY on T.

T h e o r e m 4.3. For both distributions S and T being integrable, a necessary and sufficient condition for the local components F_{2r-n} of the $F(K,K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system is that

(4.6) $$N(sX,tY) = 0,$$

for any two vector fields X and Y.

P r o o f . Let us assume that $N(sX,tY) = 0$ for any two vector fields X and Y. Therefore from Theorem 4.1, the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y. Hence $L' = 0$. This implies that the components F_{2r-n} of the $F(K,K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of the distribution S in an adapted coordinate system.

Conversely, if the components F_{2r-n} of the $F(K,K-2)$-structure are independent of these coordinates, then $L' = 0$. Therefore the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y. Hence $N(sX,tY) = 0$ for any two vector fields X and Y.

Theorem 4.4. For both distributions S and T being integrable a necessary and sufficient condition for the local components F_{2n-2r} of the $F(K,K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of T in an adapted coordinate system is that

$$N(tX,sY) = 0, \tag{4.7}$$

for any two vector fields X and Y.

The proof is similar to the proof of previous theorem.

Definition 4.1. We say that the $F(K,K-2)$-structure is integrable if

(i) the $F(K,K-2)$-structure is partially integrable;

(ii) the components F_{2r-n} of the $F(K,K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system;

(iii) the components F_{2n-2r} of the $F(K,K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of T in an adapted coordinate system.

Theorem 4.5. A necessary and sufficient condition for the $F(K,K-2)$-structure to be integrable is that

$$N(X,Y) = 0, \tag{4.8}$$

for any two vector fields X and Y.

The proof of the theorem follows from Theorems 3.7, 4.3 and 4.4.

Acknowledgement: We are grateful to Dr M.D.Upadhyay, D.Sc., for his valuable suggestions. The first author is also thankful to C.S.I.R. for financial assistance in the form of Post-Doctoral Fellowship.

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Received April 6, 1976.