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SOME PROBLEMS ON COMPLETE AND HORIZONTAL LIFTS OF F (k, k-2)-STRUCTURE

Yano and Patterson [1] studied the complete lift from a differentiable manifold M of class C^∞ to its cotangent bundle $C_{T(M)}$. The horizontal lift from M to $C_{T(M)}$ has been studied by Yano and Patterson [2]. The purpose of the present paper is to study the complete and horizontal lifts of a $F(k,k-2)$-structure, from a manifold to its cotangent bundle. Necessary notations have been given in Section 1. In Section 2, the Nijenhuis tensor of the complete lift is studied. In Section 3 the horizontal lift of a $F(k,k-2)$-structure is considered and it is found to have a $F(k,k-2)$-structure.

1. Preliminaries

Let M be a differentiable manifold of class C^∞ and dimension n and let $C_{T(M)}$ be the cotangent bundle of M and of dimension $2n$. Then $C_{T(M)}$ is also a differentiable manifold of class C^∞. Let $\mathcal{T}^r_s(M)$ denote the set of tensor fields of class C^∞ and type (r,s) in M and let $\mathcal{T}^r_s(C_{T(M)})$ denote the corresponding set of tensor fields in $C_{T(M)}$. Vector fields in M are denoted by X, Y, Z. Lie derivative with respect to X is denoted by $\mathcal{L}_X$ and the Lie product of X and Y is denoted by $[X,Y]$.

2. Nijenhuis Tensor of the complete lift of F^k (k even)

Let $F \in \mathcal{T}_1^1(M)$. Then the Nijenhuis tensor of F is given by [1]

$$(2.1)\quad N_{F,F}(X,Y) = [FX,FY] - F\,[FX,Y] - F\,[X,FY] + F^2\,[X,Y].$$

Let F be a $F(k,k-2)$ structure on M, that is

$$(2.2)\qquad F^k + F^{k-2} = 0.$$

Theorem 2.1. The Nijenhuis tensor of the complete lift of F^k vanishes if the Lie derivatives of the tensor fields F^{k-2} with respect to X and Y are both zero and F is an almost complex structure on M.

Proof. In consequence of (2.1) the Nijenhuis tensor of $(F^k)^C$ is given by

$$(2.3)\quad \begin{cases} N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = \\ = \left[(F^k)^C X^C,(F^k)^C Y^C\right] - (F^k)^C\left[(F^k)^C X^C, Y^C\right] - \\ - (F^k)^C\left[X^C,(F^k)^C Y^C\right] + (F^k)^C (F^k)^C\left[X^C,Y^C\right], \end{cases}$$

which in view of equation (2.2) yields

$$(2.4)\begin{cases} N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = \left[(-F^{k-2})^C X^C,\ (-F^{k-2}) Y^C\right] - \\ - (-F^{k-2})^C\left[(-F^{k-2})^C X^C, Y^C\right] - (-F^{k-2})^C\left[X^C,(-F^{k-2})^C Y^C\right] + \\ + (-F^{k-2})^C(-F^{k-2})^C\left[X^C,Y^C\right]. \end{cases}$$

We know from [1]

$$(F)^C X^C = (FX)^C + (\mathcal{L}_X F)^V,$$

hence

$$(F^{k-2})^C X^C = (F^{k-2}X)^C + (\mathcal{L}_X F^{k-2})^V. \tag{2.5}$$

If the Lie derivatives of $(+F^{k-2})$ with respect to X and Y both vanish i.e.

$$\mathcal{L}_X(-F^{k-2}) = 0 \quad \text{and} \quad \mathcal{L}_Y(-F^{k-2}) = 0 \tag{2.6}$$

then equation (2.4) in view of equations (2.5) and (2.6) becomes

$$\begin{cases} N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = \left[(-F^{k-2}X)^C,(-F^{k-2}Y)^C\right] - \\ - (-F^{k-2})^C \left[(-F^{k-2}X)^C,Y^C\right] - \\ - (-F^{k-2})^C \left[X^C,(-F^{k-2}Y)^C\right] - (-F^{k-2})^C(-F^{k-2})^C\left[X^C,Y^C\right]. \end{cases} \tag{2.7}$$

We know from [2] that for every $X, Y \in \mathcal{T}^0_1(M)$

$$\left[X^C,Y^C\right] = [X,Y]^C.$$

Thus (2.7) takes the form

$$\begin{cases} N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = \left[-F^{k-2}X, -F^{k-2}Y\right]^C - \\ - (-F^{k-2})^C \left[(-F^{k-2}X), Y\right]^C - \\ - (-F^{k-2})^C \left[X,(-F^{k-2}Y)\right]^C - (-F^{k-2})^C(-F^{k-2})^C[X,Y]^C. \end{cases} \tag{2.8}$$

If F is an almost complex structure on M then

$$F^2 = -I,$$

where I is unit tensor field.

If k is even then $F^{k-2} = \pm I$, consequently

$$N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = 0.$$

Hence the theorem is proved.

We shall next state and prove the following theorem.

T h e o r e m 2.2. The Nijenhuis tensor of the complete lift of F^k is equal to the complete lift of F^{k-2} if

$$\text{(i)} \qquad \mathcal{L}_X F^{k-2} = 0, \quad \mathcal{L}_Y F^{k-2} = 0$$

and

$$\text{(ii)} \qquad [X,Y]^C = 0, \quad \tilde{K}^V = 0,$$

where

$$(2.9) \qquad \tilde{K}^V \overset{\text{def}}{=\!=} \mathcal{L}_{[F^{k-2}X,Y]} F^{k-2} + \mathcal{L}_{[X,F^{k-2}Y]} F^{k-2} - \mathcal{L}_{[X,Y]} F^{2k-4}.$$

P r o o f . In consequence of (2.1), we have

$$(N_{F^{k-2},F^{k-2}}(X,Y))^C = [F^{k-2}X,F^{k-2}Y]^C - (F^{k-2}[F^{k-2}X,Y])^C - (F^{k-2}[X,F^{k-2}Y])^C + ((F^{k-2})^2[X,Y])^C,$$

which in view of (2.5) yields

$$(2.10) \qquad \begin{cases} (N_{F^{k-2},F^{k-2}}(X,Y))^C = [F^{k-2}X,F^{k-2}Y]^C - \\ - (F^{k-2})^C [F^{k-2}X,Y]^C + (\mathcal{L}_{[F^{k-2}X,Y]} F^{k-2})^V - \\ - (F^{k-2})^C [X,F^{k-2}Y]^C + (\mathcal{L}_{[X,F^{k-2}Y]} F^{k-2})^V + \\ + (F^{2k-4})^C [X,Y]^C - (\mathcal{L}_{[X,Y]} F^{2k-4})^V. \end{cases}$$

But we know that

$$(2.11)\qquad (F^{k-2})^C(F^{k-2})^C = (F^{2k-4})^C + (N_{F^{k-2},F^{k-2}})^V.$$

Hence equation (2.10) becomes

$$(2.12)\quad \begin{cases} (N_{F^{k-2},F^{k-2}}(X,Y))^C = \left[F^{k-2}X,F^{k-2}Y\right]^C - \\ - (F^{k-2})^C\left[F^{k-2}X,Y\right]^C - (F^{k-2})^C\left[X,F^{k-2}Y\right]^C + \\ + (F^{k-2})^C(F^{k-2})^C[X,Y]^C - (N_{F^{k-2},F^{k-2}})^V[X,Y]^C + \\ + (\mathcal{L}_{[F^{k-2}X,Y]}F^{k-2} + \mathcal{L}_{[X,F^{k-2}Y]}F^{k-2} - \mathcal{L}_{[X,Y]}F^{2k-4})^V. \end{cases}$$

Hence in view of equation (2.12) equation (2.8) yields

$$N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = (N_{F^{k-2},F^{k-2}}(X,Y))^C + (N_{F^{k-2},F^{k-2}})^V[X,Y]^C$$

$$- (\mathcal{L}_{[F^{k-2}X,Y]}F^{k-2} + \mathcal{L}_{[X,F^{k-2}Y]}F^{k-2} - \mathcal{L}_{[X,Y]}F^{2k-4})^V.$$

which in consequence of (2.9) gives

$$N_{(F^k)^C,(F^k)^C}(X^C,Y^C) = (N_{F^{k-2},F^{k-2}}(X,Y))^C +$$

$$+ (N_{F^{k-2},F^{k-2}})^V\left[X,Y\right]^C - \tilde{K}^V.$$

Hence the theorem follows.

T h e o r e m 2.3. The Nijenhuis tensor of the complete lift of F^k is equal to the complete lift of the Nijenhuis tensor of F^{k-2} if

(i) $\mathcal{L}_X F^{k-2} = \mathcal{L}_Y F^{k-2} = 0,$

and

(ii) $\mathcal{L}_X Y = 0, \quad \widetilde{R}^V = 0.$

The proof of the result follows from Theorem 2.3 as from $[X,Y]^C = 0$ we get $[X,Y] = 0$, i.e. $\mathcal{L}_X Y = 0$.

3. The horizontal lift of a F(k,k-2)-structure

T h e o r e m 3.1. Let $F \in \mathcal{T}^1_1(M)$ be a $F(k,k-2)$-structure on M. Then the horizontal lift F^H of F is also a $F(k,k-2)$-structure on $C_{T(M)}$.

P r o o f . For every F, $F' \in \mathcal{T}^1_1(M)$, we have [2]

(3.1) $$F^H F'^H + F'^H F^H = (FF' + F'F)^H.$$

Putting $F' = F$ in (3.1), we get

(3.2) $$(F^H)^2 = (F^2)^H.$$

Next putting $F' = F^2$ in (3.1) we get by virtue of (3.2)

(3.3) $$(F^H)^3 = (F^3)^H.$$

Further replacing F' by F^3 in (3.1) we get

$$F^H(F^3)^H + (F^3)^H F^H = (2F^4)^H$$

which in consequence of (3.3) gives

(3.4) $$(F^H)^4 = (F^4)^H.$$

Proceeding in a similar way we have

$$(F^H)^{k-2} = (F^{k-2})^H \tag{3.5}$$

$$(F^H)^k = (F^k)^H. \tag{3.6}$$

Since F is a $F(k,k-2)$-structure on M we get

$$F^k + F^{k-2} = 0.$$

Hence from (3.5) and (3.6) we get

$$(F^H)^k = (F^k)^H = (-F^{k-2})^H = (-(F^H)^{k-2})$$

or

$$(F^H)^k + (F^H)^{k-2} = 0.$$

Thus F^H is a $F(k,k-2)$-structure on $C_{T(M)}$.

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