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A GENERALIZED COMPOUND HILBERT-RIEMANN PROBLEM FOR A SYSTEM OF FUNCTIONS

The linear Hilbert-Riemann problem in the class of analytic functions was examined by J.S. Rogozhina [1] and Lu Chien Ke [3]. Some Hilbert-Riemann problems in the class of pseudoanalytic functions were considered by J. Wolska-Bochenek [2] and G.Warowna-Dorau [14]. The present author examined [15] a compound non-linear Hilbert-Riemann problem in the class of analytic functions. The aim of this paper is to solve analogous problems, both linear and non-linear, for a system of m functions $(m > 1)$. The non-linear problem will be reduced to an equivalent system of singular integral equations (see (39) below) that will subsequently be examined by using Schauder's fixed point theorem.

1. The linear problem

Let D^+ be a multiconnected domain of the (open) complex plane E whose boundary consists of disjointed closed curves $L_1, L_2, \dots, L_m$ and of the unit circle $L_0 = \{z: |z| = 1\}$. We assume that all curves L_i $(i = 0,1,\dots,m)$ are of positive direction with respect to D^+ and that $L_1,\dots,L_m$ are situated inside the circle L_0. We shall use the notation $L = \bigcup_{i=1}^{m} L_i$; $D^- = \bigcup_{i=1}^{m} D_i^-$; $\bar{S}_0 = D^- \cup L \cup D^+ \cup L_0$, where D_i^- is the domain placed inside the curve L_i.

The problem to be examined consists in finding a sectionally analytic vector $\Phi(z) = [\Phi_1(z),\dots,\Phi_m(z)]$ whose boundary values Φ^+ and Φ^- satisfy the following conditions

$$\text{(1)}\quad \begin{aligned} &\text{a)} \quad \Phi^+(t) = A(t)\,\Phi^-(t) + g(t), \qquad t \in L;\\ &\text{b)} \quad \mathrm{Re}\left[c(t_0)\,\Phi^+(t_0)\right] = H(t_0), \qquad t_0 \in L_0, \end{aligned}$$

where $A(t)$, $c(t_0)$ are given square matrices and $g(t)$, $H(t_0)$ are given vectors.

We make the following assumptions:

1° The complex non-singular matrix $A(t) = [A_{\alpha\beta}(t)]$, $(\alpha, \beta = 1,2,\dots,m)$ is defined on L and belongs to the clas $C_\mu(M_A,k_A)$ on L.*)

2° The complex diagonal matrix $c(t_0)$ is defined on L_0 and its elements $c_i(t_0)$, $i = 1,2,\dots,m$ belong to $C_\mu(M_c,k_c)$ on L_0. Moreover, $c(t_0) \neq 0$ for all $t_0 \in L_0$.

3° The real vector $H(t_0)$ is defined on L_0 and $H(t_0) \in C_\mu(M_H, k_H)$ on L_0.

4° The complex vector $g(t)$ is defined on L and $g(t) \in C_\mu(M_H,k_H)$ on L.

We introduce the following notation (see e.g. [9])

$$\text{(2)}\quad \mathrm{ind}\left[A(t)\right]_L = \frac{1}{2\pi i}\left[\ln\det A(t)\right]_L = \frac{1}{2\pi}\left[\arg\det A(t)\right]_L = \varkappa_H;$$

$$\text{(3)}\quad \mathrm{ind}\left[(c(t_0))^{-1}\cdot\overline{c(t_0)}\right]_{L_0} = \frac{1}{2\pi i}\left[\ln\det(c(t_0))^{-1}\cdot\overline{c(t_0)}\right]_{L_0} = \frac{1}{\pi}\left[\arg c(t_0)\right]_{L_0} = \varkappa_R.$$

*) We say that $A(t)$ belongs to $C_\mu(M_A,k_A)$ if each element of $A(t)$ belongs to $C_\mu(M_A,k_A)$, where $C_\mu(M_A,k_A)$ is the class of all functions whose moduli are bounded by the constant $M_A > 0$ and that satisfy Hölder's condition with the coefficient $k_A > 0$ and the exponent $\mu \in (0,1\rangle$.

The sum

$$\varkappa = \varkappa_H + \varkappa_R \tag{4}$$

will be called the index of problem (1).

2. Solution of the linear problem

It is well known (see [5], pp.368-371; [8], pp.49-52 and [9], pp. 439-442) that the general solution of the linear problem (1.a) is given by

$$Y(z) = \frac{X(z)}{2\pi i}\int_L \frac{[X^+(\tau)]^{-1} g(\tau)}{\tau - z}\, d\tau + X(z)\, P(z). \tag{5}$$

Here, $P(z)$ is a vector whose coordinates $P_i(z)$, $(i = 1,2,\ldots,m)$ are arbitrary polynomials and

$$X(z) = \left[\overset{\beta}{X}_\alpha(z)\right]; \quad \alpha, \beta = 1,2,\ldots,m \tag{6}$$

is the canonic matrix whose columns $\overset{1}{X}_\alpha(z),\ldots,\overset{m}{X}_\alpha(z)$, $(\alpha = 1,2,\ldots,m)$, forming the so-called fundamental system, are the solutions of the homogeneous problem

$$X^+(t) = A(t)\, X^-(t), \quad t \in L. \tag{7}$$

Let us note that the canonic matrix has the following properties:

a) $\det\left[\overset{\beta}{X}_\alpha(z)\right] \neq 0$ on the whole plane E;

b) The boundary values $\overset{\beta\pm}{X_\alpha}(t)$ of the elements of $X(z)$ $(\alpha, \beta = 1,2,\ldots,m)$ satisfy on L Hölder's condition with the exponent $\frac{\mu}{2}$ and the coefficient $k_x > 0$;

c) The integers $-\varkappa_{H_1}, -\varkappa_{H_2}, \ldots, -\varkappa_{H_m}$, called the partial indices of the problem (see [9], p.428) are equal to the orders at infinity of the appropriate fundamental solutions $\overset{1}{X}_\alpha(z), \overset{2}{X}_\alpha(z), \ldots, \overset{m}{X}_\alpha(z)$, $(\alpha = 1,2,\ldots,m)$. The equa-

lity $\varkappa_H = \sum_{i=1}^{m} \varkappa_{H_i}$ holds true. Assuming that $\varkappa_{H_1} \geqslant \varkappa_{H_2} \geqslant \dots$
$\dots \geqslant \varkappa_{H_s} \geqslant 0 \geqslant \varkappa_{H_{s+1}} \geqslant \dots \geqslant \varkappa_{H_m}$; $\lambda = \sum_{i=1}^{s} \varkappa_{H_i}$; $-\gamma = \sum_{i=s+1}^{m} \varkappa_{H_i}$,
we have $\varkappa_H = \lambda - \gamma$.

It follows from the results of J.S.Rogozhina [1], Lu Chien Ke [3], and J.Wolska-Bochenek [2] that if a solution of problem (1) exists, then it is given by

$$\Phi(z) = Y(z) + X(z)\,\Phi_0(z), \tag{8}$$

where $P_0(z)$ is a vector with coordinates $\Phi_{0i}(z)$, $(i = 1,2,\dots,m)$ analytic in S_0 and continuous on $\overline{S_0}$. By (1.b) we can assert that

$$\mathrm{Re}\left[c(t_0)\; X^{+}(t_0)\,\Phi_0^{+}(t_0)\right] = h(t_0); \quad t_0 \in L_0 \tag{9}$$

holds true, where

$$h(t_0) = H(t_0) - \mathrm{Re}\left[c(t_0)\; Y(t_0)\right]. \tag{10}$$

The problem (9) is called the Riemann problem and (see [8], pp.162-172) can be reduced to the Hilbert problem

$$\psi^{+}(t_0) = G(t_0)\,\psi^{-}(t_0) + g_1(t_0), \quad t_0 \in L_0, \tag{11}$$

where

$$\psi(z) = \begin{cases} \Phi_0(z) & \text{for} \quad z \in S_0 \\ \Phi_{x_0}(z) = \overline{\Phi_0\left(\frac{1}{\bar z}\right)} & \text{for} \quad z \in E - \overline{S_0}, \end{cases} \tag{12}$$

$$G(t_0) = -\left[c(t_0)\cdot X^{+}(t_0)\right]^{-1} \cdot \left[\overline{c(t_0)}\;\overline{X^{+}(t_0)}\right] \tag{13}$$

and

$$g_1(t_o) = \left[c(t_o)\, X^+(t_o)\right]^{-1} \cdot 2h(t_o). \tag{14}$$

The problems (9) and (11) are equivalent, i.e. either both or none of them have a solution. If a vector $\tilde{\psi}(z)$ is a solution of problem (11), then a solution of the Riemann problem (9) can be found from the formula

$$\Omega(z) = \frac{1}{2}\left[\tilde{\psi}(z) + \overline{\tilde{\psi}\left(\frac{1}{\bar{z}}\right)}\right]. \tag{15}$$

It is known that the non-homogeneous Hilbert problem (11) has a solution bounded at infinity if and only if the condition

$$\int_L q(t)\left[X^+(t)\right]^{-1} g_1(t)\, dt = 0 \tag{16}$$

is fulfilled. Here, $q(t)$ is a vector whose coordinates $q_\alpha(t)$ (note that $q_\alpha(t) \equiv 0 \iff \alpha < 0$, $\alpha = -\varkappa_{R_1}-2, -\varkappa_{R_2}-2, \ldots, -\varkappa_{R_m}-2$) are arbitrary polynomials of order α and $X(z)$ is the canonic matrix of the solutions of the homogeneous problem

$$X^+(t_o) = G(t_o)\, X^-(t_o), \quad t_o \in L_o. \tag{17}$$

If condition (16) is satisfied, then the vector

$$\omega(z) = \frac{X(z)}{2\pi i}\int_{L_o} \frac{\left[X^+(\tau)\right]^{-1} g_1(\tau)\, d\tau}{\tau - z} \tag{18}$$

is the unique solution of problem (9). The general solution of problem (9) is given by

$$(19)\qquad \Phi_o(z) = P_o(z)\, X(z) + \frac{1}{2}\left[\omega(z) + \overline{\omega\left(\frac{1}{\bar z}\right)}\right],$$

where the coordinates $P_{oi}(z)$ of $P_o(z)$ - vector are polynomials of the form $P_{oi}(z) = c_i^{(o)} z^{\varkappa_{R_i}} + c_i^{(1)} z^{\varkappa_{R_i}-1} + \dots + c_i^{(\varkappa_R)}$ $(i = 1,2,\dots,\varkappa_{R_i})$ whose coefficients $c_i^{(j)}$ satisfy the conditions $c_i^{(j)} = \bar c_i^{(\varkappa_{R_i}-j)}$; $j = 0,1,2,\dots,\varkappa_R$.

From the considerations above it results that the followwing theorem is valid[*)]

T h e o r e m 1. If $\varkappa \geqslant 0$ and if assumptions $1^o - 4^o$ are satisfied, then vector (8) (where $Y(z)$ and $\Phi_o(z)$ are given by (5) and (19) respectively) is a solution of problem (1).

3. <u>The compound non-linear problem</u>

Further generalization of the mixed Hilbert-Riemann problem is the following problem:

To find a vector $\Phi(z) = \left[\Phi_1(z),\Phi_2(z),\dots,\Phi_m(z)\right]$, sectionally analytic in D^+ and D^-, whose boundary values satisfy the conditions

$$(20)\quad \begin{cases} \text{a)}\quad \Phi^+(t) = A(t)\,\Phi^-(t) + F\left[t,\overline{\Phi^+(t)},\Phi^+(t),\overline{\Phi^-(t)},\ \Phi^-(t)\right], \quad t \in L \\ \text{b)}\quad \mathrm{Re}\left[c(t_o)\,\Phi^+(t_o)\right] = H(t_o), \quad t_o \in L_o. \end{cases}$$

We retain the assumptions 1^o - 3^o and we make the following assumption:

5^o. The vector $F(t,u_1,\dots,u_{4m}) = \left[F_1(t,u_1,\dots,u_{4m}),\dots,F_m(t,u_1,\dots,u_{4m})\right]$ is defined in the set $\{t \in L,\ |u_i| < R\}$,

*) This theorem is a generalization of Theorem 1 in [15].

($i=1,2,\dots,m$; R being a positive number) and satisfies in this set the inequalities

$$\left|F(t,\, u_1,\dots,u_{4m})\right| \leq M_F\, \Big(1 + \sum_{i=1}^{4m} |u_i|\Big),$$

$$\left|F(t,u_1,\dots,u_{4m}) - F(t',\, u'_1,\dots,u'_{4m})\right| \leq$$
$$\leq k_F \left\{ |t-t'|^{\mu} + \sum_{i=1}^{4m} |u_i - u'_i| \right\},$$

where $M_F, k_F > 0$; $\mu \in (0,1\rangle$.

Let us suppose for a moment that the vector $F(t,u_1,\dots,u_{4m})$ is given. In this case, by the considerations concerning the linear problem (1) above, one can assert that the solution of the problem

$$(21)\qquad \begin{cases} \Phi^+(t) = A(t)\,\Phi^-(t) + F(t,u_1,\dots,u_{4m}), & t \in L \\ \mathrm{Re}\left[c(t_0)\,\Phi^+(t_0)\right] = H(t_0), & t_0 \in L_0 \end{cases}$$

is determined by[*)]

$$(22)\qquad \Phi(z) = w_0(z) + X(z)\, w_2(z),$$

where

$$(23)\qquad w_0(z) = \frac{X(z)}{2\pi i}\int_L \frac{\left[X^+(\tau)\right]^{-1} \cdot F(\tau,u_1,\dots,u_{4m})}{\tau - z}\, d\tau + $$
$$+ X(z)\, P(z)$$

---

*) We assume that the index $\varkappa$ is non-negative and that $\varkappa_{H_i} \geq 0$ for $i = 1,2,\dots,m$. In the opposite case one should make additional assumptions of type (16) concerning the unknowns of the problem.

and $w_2(z)$ is a solution of the following Riemann problem

$$(24)\qquad \mathrm{Re}\left[c(t_o)X^+(t_o)\, w_2(t_o)\right] = H(t_o) - \mathrm{Re}\left[c(t_o)w_o(t_o)\right]$$

given by

$$(25)\qquad w_2(z) = P_o(z)\, X(z) + \frac{1}{2}\left[w_3(z) + \overset{*}{w}_3(z)\right]$$

with

$$(26)\qquad w_3(z) = \frac{X(z)}{2\pi i}\int_L \frac{\left[X^+(z)\right]^{-1} w_4(\tau)}{\tau - z}\, d\tau;$$

$$(27)\qquad w_4(t_o) = H(t_o) - \mathrm{Re}\left[c(t_o)\, w_o(t_o)\right].$$

4. Reduction of the non-linear problem to a system of integral equations

Vector (22) has been constructed above by a formal use of the results of papers [1]-[3] and [15] and of section 2 in this paper. Now, we shall find sufficient conditions for the existence of a solution of problem (20) in the form (22).

Let us note that the vector $w_2(t)$ is continuous on L and the coordinates

$$(28)\qquad \Phi^+_\alpha(t) = u_\alpha(t),\quad \Phi^-_\alpha(t) = u_{\alpha+2m}(t),\qquad \alpha = 1,2,\dots,m$$

of the boundary values on L of the vector $\Phi(z)$ can be found from (22) by using the Sochocki-Plemelj formulas (cf. [4], p. 7 and 123).

Let us also observe that the boundary values $u_1(t),\dots$ $\dots u_m(t)$, $u_{2m+1}(t)$, ..., $u_{3m}(t)$ satisfy the following system of singular integral equations (see [6]):

(29) $$u_n(t) = f_n(t) + F_n^*(t) + \int_L \frac{F_n^{**}(t,\tau)}{\tau - t}\, d\tau$$

$(n=1,2,\dots,m,2m+1,\dots,3m)$, where

(30) $$f_n(t) = \begin{cases} \sum_{\beta=1}^{m} \overset{\beta}{X}{}_n^{+}(t)\left[P_{0\beta}(t) + \overset{\beta}{w}_2(t)\right], & n=1,2,\dots,m \\ \sum_{\beta=1}^{m} \overset{\beta}{X}{}_{n-2m}^{-}(t)\left[P_{0\beta}(t) + \overset{\beta}{w}_2(t)\right], & n=2m+1,\dots,3m, \end{cases}$$

(31) $$F_n^*(t) = \begin{cases} \frac{1}{2} F_n(t,u_1,\dots,u_{4m}), & n=1,2,\dots,m \\ -\frac{1}{2}\sum_{r=1}^{m} A_{r,n-2m}(t)F_{n-2m}(t,u_1,\dots,u_{4m}), & n=2m+1,\dots,4m; \end{cases}$$

(32) $$F_n^{**}(t,\tau) = \begin{cases} \frac{1}{2\pi i}\sum_{r,\beta=1}^{m} \overset{\beta}{X}{}_n^{+}(t)\, \overset{\beta}{x}{}_r^{+}(\tau)F_n(\tau,u_1,\dots,u_{4m}), & n=1,2,\dots,m \\ \frac{1}{2\pi i}\sum_{r,\beta=1}^{m} \overset{\beta}{X}{}_{n-2m}^{-}(t)\overset{\beta}{x}{}_r^{+}(\tau)F_{n-2m}(\tau,u_1,\dots,u_{4m}), & n=2m+1,\dots,3m. \end{cases}$$

and $\overset{\beta}{x}{}_r^{+}$ are the elements of the inverse matrix of the matrix $X^+(t)$.

By using Lemma 1 in [15], system (30) is reduced to the equivalent system of singular integral equations of the form

(33) $$w_1(t) = f(t) + F^*(t) + \int_L \frac{F^{**}(t,\tau)}{\tau - t}\, d\tau,$$

where

$$(34)\quad w_1(t) = \left[u_1(t),\dots,u_{4m}(t)\right];\ f(t) = \left[f_1(t),\dots,f_{4m}(t)\right];$$

$$F^*(t)=\left[F_1^*(t),\dots,F_{4m}^*(t)\right];\ F^{**}(t,\tau)=\left[F_1^{**}(t,\tau),\dots,F_{4m}^{**}(t,\tau)\right]$$

are vectors with coordinates given by

$$(35)\quad f_v(t) = \begin{cases} \sum_{\beta=1}^{m} \overset{\beta}{X}{}_v^+(t)\left[P_{0\beta}(t) + \overset{\beta}{w}_2(t)\right], & v=1,2,\dots,m,\\ \sum_{\beta=1}^{m} \overline{\overset{\beta}{X}{}_{v-m}^+(t)}\ \left[\overline{P_{0\beta}(t)} + \overline{\overset{\beta}{w}_2(t)}\right], & v=m+1,\dots,2m,\\ \sum_{\beta=1}^{m} \overset{\beta}{X}{}_{v-2m}^-(t)\left[P_{0\beta}(t)+w_2(\overset{\beta}{t})\right] & v=2m+1,\dots,3m,\\ \sum_{\beta=1}^{m} \overline{\overset{\beta}{X}{}_{v-3m}^-(t)}\left[\overline{P_{0\beta}(t)}+\overline{w_2(\overset{\beta}{t})}\right] & v=3m+1,\dots,4m, \end{cases}$$

$$(36)\quad F_v^*(t) = \begin{cases} \frac{1}{2} F_v\ (t,u_1,\dots,u_{4m}), & v=1,2,3,\dots,m,\\ \frac{1}{2}\ \overline{F_{v-m}\ (t,u_1,\dots,u_{4m})}, & v=m+1,\dots,2m,\\ -\frac{1}{2}\sum_{r=1}^{m} A_{r,r-2m}(t)\ F_{v-2m}(t,u_1,\dots,u_{4m}), & v=2m+1,\dots,3m,\\ -\frac{1}{2}\sum_{r=1}^{m} \overline{A_{r,r-3m}(t)}\ \overline{F_{v-3m}\ (t,u_1,\dots,u_{4m})}, & v=3m+1,\dots,4m, \end{cases}$$

$$
(38)\ F_v^{**}(t,\tau) = \begin{cases}
\dfrac{1}{2\pi i} \displaystyle\sum_{r,\beta=1}^{m} \overset{\beta}{X}{}_v^{+}(t)\, \overset{\beta}{x}{}_r^{+}(\tau)\, F_v(\tau,u_1,\dots,u_{4m}), & v=1,2,\dots,m \\[2ex]
\dfrac{1}{2\pi i} \displaystyle\sum_{r,\beta=1}^{m} \overline{\overset{\beta}{X}{}_{v-m}^{+}(t)}\, \overline{\overset{\beta}{x}{}_r^{+}(\tau)}\, \overline{F_{v-m}(\tau,u_1,\dots,u_{4m})}\, e^{2i\nu(t,\tau)}, & v=m+1,\dots,2m, \\[2ex]
\dfrac{1}{2\pi i} \displaystyle\sum_{r,\beta=1}^{m} \overset{\beta}{X}{}_{v-2m}^{-}(t)\, \overset{\beta}{x}{}_r^{+}(\tau)\, F_{v-2m}(\tau,u_1,\dots,u_{4m}), & v=2m+1,\dots,3m, \\[2ex]
\dfrac{1}{2\pi i} \displaystyle\sum_{r,\beta=1}^{m} \overline{\overset{\beta}{X}{}_{v-3m}^{-}(t)}\, \overline{\overset{\beta}{x}{}_r^{+}(\tau)}\, \overline{F_{v-3m}(\tau,u_1,\dots,u_{4m})}\cdot e^{2i\nu(t,\tau)}, & v=3m+1,\dots,4m,
\end{cases}
$$

respectively, where $\nu(t,\tau) = \arg(t-\tau)$.

Hence, problem (20) has been reduced to the following equivalent system of singular integral equations

$$
(39)\ \begin{cases}
w_1(t) = f(t) + F^*(t) + \displaystyle\int_L \frac{F^{**}(t,\tau)\,d\tau}{\tau - t} \equiv T_1(w_1,w_2), \\[2ex]
w_2(z) = P_0(z)\,X(z) + \frac{1}{2}\left[w_3(z) + w_3^*(z)\right] \equiv T_2(w_3), \\[2ex]
w_3(z) = \dfrac{X(z)}{2\,i} \displaystyle\int_L \frac{\left[(X^+(\tau)\right]^{-1} w_4(\tau)\,d\tau}{\tau - z} \equiv T_3(w_4), \\[2ex]
w_4(t_0) = H(t_0) - \operatorname{Re}\left[c(t_0)w_0(t_0)\right] \equiv T_4(w_0),
\end{cases}
$$

where $w_0(t_0)$ is given by formula (23) with $z = t_0$, whence $T_4(w_0) = T_4(w_1)$. Therefore, if problem (20) has a solution of the form (22), then system of integral equations (39) possesses a solution and vice versa.

5. Examination of the system of integral equations and solution of the non-linear problem

In this section, system (39) will be examined by using Schauder's fixed-point theorem. To this end let us consider the space Λ of all points $p = (w_1, w_2, w_3, w_4)$ where the vectors w_1, w_2 and w_3 are defined and bounded in S_0 and continuous in D^+ and D^- separately, and the vector w_4 is defined and continuous on L_0. Note that the vector w_1 has a "jump" on L attaining there the values $u_\nu(t)$, $(\nu = 1,2,\ldots,4m)$ and the values on L of the vectors w_2 and w_3 can be found from formulas (25) and (26) with taking into account the values on L of the canonical solution $X(z)$ of problem (7).

We introduce the addition of two points of Λ and the product of a point and a real number in the usual way, and we define the norm $\|p\|$ of a point p and the distance $\varrho(p^{(1)}, p^{(2)})$ of two points $p^{(1)}$ and $p^{(2)}$ by the formulas

$$(40)\qquad \begin{cases} \|p\| = \sum_{i=1}^{3} \sup_{\bar{S}_0} |w_i| + \sup_{L_0} |w_4| \\ \varrho(p^{(1)}, p^{(2)}) = \|p^{(1)} - p^{(2)}\| \end{cases}$$

respectively. It is easily observed that Λ is a Banach space.

We now consider in the space Λ the set V of all points p whose coordinates w_1, w_2, w_3, w_3 satisfy

$$(41)\qquad w_i \in C_{\frac{\mu}{2}}(\varrho_i, \varphi_i), \quad (i=1,2,3,4),$$

where ϱ_1 and φ_1 are arbitrarily fixed positive numbers, and ϱ_i, φ_i $(i = 2,3,4)$ are also positive and depend on ϱ_1, φ_1. Evidently, V is a closed convex set.

In view of the form of system (39) we map the set V onto a set $V^* \subset \Lambda$ by the transformation

$$(42)\qquad \begin{cases} \omega_1(t) = T_1(w_1, w_2), \\ \omega_2(z) = T_2(w_3), \\ \omega_3(z) = T_3(w_4), \\ \omega_4(z) = T_4(w_1). \end{cases}$$

L e m m a 1. A sufficient condition for the inclusion $V^* \subset V$ is that the system of inequalities

$$(43)\qquad \begin{cases} A_1M_F + a_1M_F + a_2M_F\varrho_1 + a_3k_F + a_4k_F\varphi_1 + A_2\varrho_2 \leq \varrho_1, \\ a_{14}M_P + a_{15}\varrho_3 \leq \varrho_2, \\ a_{10}\varrho_4 + a_{11}\varphi_4 \leq \varrho_3, \\ a_0 + a_5M_P + a_6M_F + a_7M_F\varrho_1 + a_8k_F + a_9k_F\varphi_1 \leq \varrho_4, \\ B_1M_P + B_3k_P + b_1M_F + b_2M_F\varrho_1 + b_3k_F + b_4k_F\varphi_1 + B_2\varrho_2 + \\ \qquad + B_4\varphi_2 \leq \varphi_1, \\ a'_{14}M_P + a'_{15}k_P + a'_{13}\varphi_3 \leq \varphi_2, \\ a_{12}\varrho_4 + a_{13}\varphi_4 \leq \varphi_3, \\ b_0 + b_5M_P + b_5k_P + b_7M_F + b_8M_F\varrho_1 + b_9k_F + b_{10}k_F\varphi_1 \leq \varphi_4, \end{cases}$$

is valid, where the constants $A_1, A_2, B_1, \ldots, B_4$, a_i $(i=1,2,\ldots,15)$ and b_j $(j=1,2,\ldots,9)$ are independent of F_ν $(\nu=1,2,\ldots,4m)$ and P_α $(\alpha=1,2,\ldots,m)$.

P r o o f . Inequalities (43) result from assumptions $1^o - 3^o$, 5^o, (41), definition (40) and the estimates given in [6].

It is easily seen that system (43) holds true if the coefficients M_F and k_F satisfy the conditions

$$(44)\quad \begin{cases} M_F < \left[a_2 + (a_7 a_{10} + a_{11} b_8)\, a_{15} A_1\right]^{-1} \\ k_F < \left[b_4 + B_2 a_{15}(a_9 a_{10} + a_{11} a_{10}) + B_4 a_{13}(b_{10} a_{13} + a_9 a_{12})\right]^{-1}. \end{cases}$$

L e m m a 2. The set V^* is compact.

The validity of Lemma 2 results from (43) and from Arzela's theorem.

L e m m a 3. Transformation (42) is continuous in the space Λ.

Proof is analogous to that in [4].

Thus, all assumptions of Schauder's fixed point theorem (see e.g. [4], vol. II, pp.16-26) are satisfied and hence we can conclude that there exists at least one fixed point $p^o = (w_1^o, w_2^o, w_3^o, w_4^o)$ of operation (42) that is a solution of system (39). Using the coordinates of p^o and relations (22)-(27), one can find a solution of the compound non-linear problem (20).

As a result we can conclude with the following theorem.

T h e o r e m 2. If the vectors $H(t_o)$ and $F(t,u_1,\ldots,u_{4m})$ and the matrices $A(t)$ and $c(t_o)$ satisfy the assumptions 3^o, 5^o, 1^o and 2^o respectively, and if the constants M_F and k_F are so small that inequalities (44) hold good, then there is a Hölder - continuous vector $\Phi(z) = (\Phi_1(z), \ldots, \Phi_m(z))$ (with the Hőlder exponent not greater than $\frac{\mu}{2}$), sectionally analytic in the domains D^+ and D , whose boundary values satisfy conditions (20).

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