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GENERAL TIME SETS

Introduction

The aim of this paper is a comparison of two abstract notions of time axis - one used in [1] for the need of the characterization of machines by means of sets of computation and the other usually applied in the theory of dynamic processes (cf. [2]). In [1] computations were treated as runs of some changes in time and it turned out that to define sets of machine computations it suffices to consider only few properties of the time axis. Namely, it suffices to interprete the time axis as a triple $(T, \prec, \theta)$, where T is the set of moments $\prec$ is the order induced upon the set of moments by the relation "not later than", and θ defines a family of translations owing to which it is possible to compare runs in various time intervals. In the theory of dynamic processes the time axis is understood as a pair $(T, \oplus)$, where T is the set of moments and $\oplus$ is "addition" of moments such that $(T, \oplus)$ is a monoid having some properties sufficient for describing dynamic processes.

The main result of this paper is a theorem stating that the time set of [2] always generates a generalized time set $(T, \prec, \theta)$. Hence having the set of moments and the addition of moments one can always introduce a natural order $\prec$ and define a function of translations θ. Also it is shown that for every generalized time set $(T, \preccurlyeq, \theta)$ one can define a monoid $(T, \oplus)$ generating $(T, \prec, \theta)$. However, this monoid $(T, \oplus)$ has, generally speaking, fewer properties that that used in the theory of dynamic processes.

1. Monoid of movements

Let $(T, \leqslant, \theta)$ be a general time set. Let $\mathcal{T} = \theta(T)$ and let $\circ$ be the composition of functions. By 0 we denote the first element of $(T, \leqslant)$ (see [1]).

Theorem 1.1. The pair $(\mathcal{T}, \circ)$ is a semigroup with an identity.

Proof. If $\alpha, \beta \in \mathcal{T}$, then $\alpha \circ \beta \in \mathcal{T}$ by Definition 1.2 in [1]. Moreover, the identity $\iota: T \to T$ belongs to $\mathcal{T}$ by Theorem 1.1 in [1].

Definition 1.1. Let $a, b \in T$, $\alpha, \beta \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$. We define an ordering relation $\leqslant$ in $\mathcal{T}$

$$(\alpha \leqslant \beta) \Longleftrightarrow (a \leqslant b). \tag{1.1}$$

Definition 1.1 is correct, because by Definition 1.2 in [1] for any $t \in T$ there exists exactly one $\varphi \in \mathcal{T}$ such that $\varphi(0) = t$.

The ordered set $(T, \leqslant)$ is similar to $(\mathcal{T}, \leqslant)$. The function $\theta: T \to \mathcal{T}$ is a similarity mapping.

Definition 1.2. The triple $(\mathcal{T}, \circ, \leqslant)$ is called the monoid of movements of a general time set $(T, \leqslant, \theta)$ or, shortly, a monoid of movements.

Theorem 1.2. If $(\mathcal{T}, \circ, \leqslant)$ is a monoid of movements, then for any $\alpha, \beta \in \mathcal{T}$ we have

$$(\alpha \leqslant \beta) \Longleftrightarrow \exists_{\gamma \in \mathcal{T}} (\alpha \circ \gamma = \beta). \tag{1.2}$$

Proof. Let $\alpha(0) = a$, $\beta(0) = b$. If $a \leqslant b$, then $b \in T_a$ and since $\alpha(T) = T_a$, there exists $c \in T$ such that $\alpha(c) = b$. Let $\gamma \in \mathcal{T}$ and $\gamma(0) = c$. We have $\alpha \circ \gamma \in \mathcal{T}$, and because $(\alpha \circ \gamma)(0) = \alpha[\gamma(0)] = \alpha(c) = b$, we obtain $\alpha \circ \gamma = \beta$. If $\alpha, \beta, \gamma \in \mathcal{T}$ and $\alpha \circ \gamma = \beta$, then $a = \alpha(0) \leqslant \alpha(c) = (\alpha \circ \gamma)(0) = \beta(0) = b$, thus $\alpha \leqslant \beta$.

R e m a r k . Since $(\alpha \leqslant \beta)$ iff $\beta(T) \subset \alpha(T)$, the composition $\alpha^{-1} \circ \beta$ is determined iff $\alpha < \beta$. Hence, if $\alpha \circ \gamma = \beta$ then $\gamma = \alpha^{-1} \circ \beta$.

T h e o r e m 1.3. If $(\mathcal{T}, \circ, <)$ is a monoid of movements, then for any $\alpha, \beta \in \mathcal{T}$ we have

$$(1.3) \qquad (\alpha \leqslant \beta) \Longleftrightarrow (\alpha^{-1} \circ \beta) \in \mathcal{T}.$$

Proof is obvious.

T h e o r e m 1.4. The identity ι is the first element of $(\mathcal{T}, <)$.

Proof is obvious.

T h e o r e m 1.5. If $(\mathcal{T}, \circ, <)$ is a monoid of movements, then for any $\alpha, \beta, \gamma \in \mathcal{T}$ we have

$$(1.4) \qquad \alpha < \alpha \circ \beta$$

$$(1.5) \qquad (\alpha \leqslant \beta) \longleftrightarrow (\gamma \circ \alpha < \gamma \circ \beta)$$

$$(1.6) \qquad (\gamma < \alpha < \beta) \longleftrightarrow (\gamma^{-1} \circ \alpha < \gamma^{-1} \circ \beta).$$

P r o o f . Let $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$. Because $\alpha(0) < \alpha(b)$ ($\alpha : T \to T_a$ is a similarity mapping) and $(\alpha \circ \beta)(0) = \alpha(b)$, we have $\alpha < \alpha \circ \beta$. Thus we have proved (1.4). If $\alpha \circ \beta$, then $a < b$ and $\gamma(a) < \gamma(b)$ ($\gamma : T \to T_c$ is a similarity mapping). Because $(\gamma \circ \alpha)(0) = \gamma(a)$ and $(\gamma \circ \beta)(0) = \gamma(b)$, we get $\gamma \circ \alpha < \gamma \circ \beta$. If $\gamma \circ \alpha < \gamma \circ \beta$, then $\gamma(a) < \gamma(b)$. Hence $a < b$ and $\alpha < \beta$. We have proved (1.5). If $\gamma \leqslant \alpha < \beta$, then $\gamma^{-1} \circ \alpha \in \mathcal{T}$ and $\gamma^{-1} \circ \beta \in \mathcal{T}$ by Theorem 1.3. The relation $a < b$ implies $\gamma^{-1}(a) < \gamma^{-1}(b)$ ($\gamma^{-1} : T_c \to T$ is a similarity mapping) and $(\gamma^{-1} \circ \alpha)(0) = \gamma^{-1}(a)$, $(\gamma^{-1} \circ \beta)(0) = \gamma^{-1}(b)$. Hence $\gamma^{-1} \circ \alpha < \gamma^{-1} \circ \beta$. If $\gamma^{-1} \circ \alpha \leqslant \gamma^{-1} \circ \beta$, then $\gamma^{-1} \circ \alpha \in \mathcal{T}$ and $\gamma^{-1} \circ \beta \in \mathcal{T}$, thus $\gamma < \alpha$ and $\gamma < \beta$ by Theorem 1.3. Moreover, if $\gamma^{-1} \circ \alpha < \gamma^{-1} \circ \beta$, then $\gamma^{-1}(a) < \gamma^{-1}(b)$, thus $a < b$ and $\alpha < \beta$. Hence $\gamma < \alpha < \beta$. We have proved (1.6).

T h e o r e m 1.6. Let $(\mathcal{T}, \circ, <)$ be a monoid of movements. For any $\alpha, \beta, \gamma \in \mathcal{T}$ if $\gamma < \beta < \alpha$ then

$$(1.7) \qquad (\gamma^{-1} \circ \beta) \circ (\beta^{-1} \circ \alpha) = \gamma^{-1} \circ \alpha .$$

P r o o f . If $\alpha, \beta, \gamma \in \mathcal{T}$ and $\gamma < \beta < \alpha$, then all the compositions in (1.7) are determined and belong to $\mathcal{T}$. If $\alpha(0) = a$ and $\beta^{-1}(a) = d$, then $[(\gamma^{-1} \circ \beta) \circ (\beta^{-1} \circ \alpha)](0) = (\gamma^{-1} \circ \beta)(d) = \gamma^{-1}(a) = (\gamma^{-1} \circ \alpha)(0)$. We have proved (1.7).

2. Time monoid

Let $(\mathcal{T}, \circ, <)$ be the monoid of movements of a general time set $(T, <, \theta)$.

D e f i n i t i o n 2.1. Let $a,b,c \in T$, $\alpha, \beta, \gamma \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$. We define an operation $\oplus$ in the set T as follows:

$$(2.1) \qquad (a \oplus b = c) \Leftrightarrow (\alpha \circ \beta = \gamma).$$

T h e o r e m 2.1. If $(T, <, \theta)$ is a general time set, then the monoid of movements $(\mathcal{T}, \circ, <)$ and the triple $(T, \oplus, <)$ are isomorphic with respect to $\oplus$ and $<$.

P r o o f . By (2.1) and (1.1) the function $\theta : T \to \mathcal{T}$ satisfies the conditions

$$(2.2) \qquad \forall_{a,b \in T} \left[\theta(a \oplus b) = \theta(a) \circ \theta(b)\right],$$

$$(2.3) \qquad \forall_{a,b \in T} \left[(a < b) \leftrightarrow \theta(a) < \theta(b)\right].$$

Thus θ is an isomorphism with respect to $\oplus$ and $<$.

D e f i n i t i o n 2.2. The triple $(T, \oplus, <)$ is called a time monoid.

T h e o r e m 2.2. If $(T, \oplus, <)$ is a time monoid, then for any $a,b \in T$

$$(2.4)\qquad (a < b) \Longleftrightarrow \exists_{c \in T} (a \oplus c = b).$$

Proof is obvious.

T h e o r e m 2.3. The first element 0 of $(T, \leqslant)$ is the identity of the time monoid $(T, \oplus, \leqslant)$.

Proof is obvious.

T h e o r e m 2.4. If $(T, \oplus, \leqslant)$ is a time monoid, then for any $a,b \in T$ there exists exactly one element $c \in T$ such that $a \oplus c = b$ iff $a \leqslant b$.

P r o o f . Let $\alpha, \beta, \gamma \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$. We have $a \oplus c = b$ iff $\alpha \circ \gamma = \beta$. The equality $\alpha \circ \gamma = \beta$ is satisfied iff $\gamma = \alpha^{-1} \circ \beta$ (see the Remark after Theorem 1.2). Hence the element c defined by $(\alpha^{-1} \circ \beta)(0) = c$ is the unique element such that $a \oplus c = b$.

D e f i n i t i o n 2.3. Let $(T, \oplus, \leqslant)$ is a time monoid and a,b,c T. We define an operation $\ominus$ as follows:

$$(2.5)\qquad (a \ominus b = c) \Longleftrightarrow (a \oplus c = b).$$

T h e o r e m 2.5. If $a,b,c \in T$, $\alpha, \beta, \gamma \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$, then

$$(2.6)\qquad (a \ominus b = c) \Longleftrightarrow (\alpha^{-1} \circ \beta = \gamma).$$

Proof is obvious.

T h e o r e m 2.6. If $a,b,c \in T$ and $c \leqslant b \leqslant a$, then

$$(2.7)\qquad (c \ominus b) \oplus (b \ominus a) = c \ominus a.$$

Proof is obvious by Theorem 1.6 and Theorem 2.5.

T h e o r e m 2.7. If a,b,c T and $a \oplus b \leqslant c$, then

$$(2.8)\qquad (a \oplus b) \ominus c = b \ominus (a \ominus c).$$

P r o o f . Let $\alpha, \beta, \gamma, \delta \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$, $\delta(0) = d$. If $(a \oplus b) \ominus c = d$, then by Theorem 2.5 we have $(\alpha \circ \beta)^{-1} \circ \gamma = \delta$. Hence $\delta = (\beta^{-1} \circ \alpha^{-1}) \circ \gamma = \beta^{-1} \circ (\alpha^{-1} \circ \gamma)$. Thus $d = b \ominus (a \ominus c)$ and we have (2.8).

T h e o r e m 2.8. If $a,b,c \in T$ and $c \leqslant a \oplus b$, then

$$c \ominus (a \oplus b) = (c \ominus a) \oplus b. \tag{2.9}$$

P r o o f . Let $\alpha, \beta, \gamma, \delta \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$, $\delta(0) = d$. If $c \ominus (a \oplus b) = d$, then $\gamma^{-1} \circ (\alpha \circ \beta) = \delta$ by Theorem 2.5. Hence $\delta = (\gamma^{-1} \circ \alpha) \circ \beta$, consequently $d = (c \ominus a) \oplus b$, thus we have (2.9).

T h e o r e m 2.9. If $(T, \oplus, \leqslant)$ is a time monoid, then for any $a,b,c \in T$ we have

$$a \leqslant a \oplus b, \tag{2.10}$$

$$(a \leqslant b) \leftrightarrow (c \oplus a \leqslant c \oplus b), \tag{2.11}$$

$$(c \leqslant a \leqslant b) \leftrightarrow (c \ominus a \leqslant c \ominus b). \tag{2.12}$$

Proof is obvious by Theorem 1.5 and Theorem 2.1.

3. Generators of general time sets

T h e o r e m 3.1. If $(T, \oplus, \leqslant)$ is a time monoid, then $(T, \oplus)$ is a monoid such that

$$\forall_{a,b,c \in T} \; (c \oplus a = c \oplus b) \Longrightarrow (a = b), \tag{3.1}$$

$$\forall_{a,b \in T} \; (a \oplus b) = 0 \Longrightarrow (b = 0), \tag{3.2}$$

$$\forall_{a,b \in T} \; \exists_{c \in T} \; (a \oplus c = b) \vee (b \oplus c = a). \tag{3.3}$$

P r o o f . If $c \oplus a = c \oplus b$, then $c \leq c \oplus a$, $c \leq c \oplus b$ by (2.10) and $c \ominus (c \oplus a) = a$, $c \ominus (c \oplus b) = b$ by Theorem 2.8 and Theorem 2.5. Thus we have (3.1). If $a \oplus b = 0$, then $a \leq 0$, and because 0 is the first element of $(T, \leq)$, we get $a = 0$. Moreover, $0 \oplus b = b$. Hence we have (3.2). If $a, b \in T$, then $a \leq b$ or $b \leq a$, thus there exists $c \in T$ such that $a \oplus c = b$ or $b \oplus c = a$ by Theorem 2.2. We have proved (3.3).

D e f i n i t i o n 3.1. A semigroup $(T, \oplus)$ is called a generator of a general time set $(T, \leq, \theta)$ iff there exists in T an ordering relation $\leq$ such that the triple $(T, \oplus, \leq)$ is a time monoid isomorphic with respect to $\oplus$ and $\leq$ to the monoid of movements $(\mathcal{T}, \circ, \leq)$ of the general time set $(T, \leq, \theta)$.

D e f i n i t i o n 3.2. A semigroup $(T, \oplus)$ is called a time generating iff there exists a general time set $(T, \leq, \theta)$ such that $(T, \oplus)$ is a generator of $(T, \leq, \theta)$.

T h e o r e m 3.2. A semigroup $(T, \oplus)$ is a time generating iff $(T, \oplus)$ satisfies the conditions (3.1), (3.2) and (3.3).

P r o o f . Let $(T, \oplus)$ be a monoid such that the conditions (3.1), (3.2) and (3.3) are satisfied. We define in T a relation $\leq$ as follows:

$$(3.4) \qquad (a \leq b) \Longleftrightarrow \exists_{c \in T} (a \oplus c = b).$$

The relation $\leq$ is reflexive because $a \oplus 0 = a$ for any $a \in T$ (0 is an identity of $(T, \oplus)$). If $a \leq b$ and $b \leq a$, then there exist $c_1, c_2 \in T$ such that $a \oplus c_1 = b$ and $b \oplus c_2 = a$, thus $(a \oplus c_1) \oplus c_2 = a$. Hence $a \oplus (c_1 \oplus c_2) = a \oplus 0$ and $c_1 \oplus c_2 = 0$ by (3.1). Thus $c_2 = 0$ by (3.2) and because $b \oplus c_2 = a$, we have $b = a$. The relation $\leq$ is antisymmetric. If $a \leq b$ and $b \leq c$, then there exist $d_1, d_2 \in T$ such that $a \oplus d_1 = b$ and $b \oplus d_2 = c$. Hence $(a \oplus d_1) \oplus d_2 = c$, so that $a \oplus (d_1 \oplus d_2) = c$, consequently $a < c$. The relation is transitive. For any $a, b \in T$ we have $a < b$ or $b < a$ by (3.3). The relation is connective. Hence the relation $<$ is a linear ordering relation in T. For any $a \in T$ let α_a be a mapping $\alpha_a : T \to T$ such that

$$(3.5) \qquad \forall_{t \in T} \ \alpha_a(t) = a \oplus t.$$

Let $\mathcal{T} = \{\alpha : \alpha = \alpha_a \wedge a \in T\}$ and let θ be a function $\theta : T \to \mathcal{T}$ such that

$$(3.6) \qquad \forall_{a \in T} \ \theta(a) = \alpha_a.$$

Let $T_a = \{t \in T : a < t\}$. Because $t < a \oplus t$, we have $\alpha(T) = T_a$. Any mapping $\alpha \in \mathcal{T}$ is 1-1, because if $a \oplus x = a \oplus x'$, then $x = x'$ by (3.1). Any mapping $\alpha \in \mathcal{T}$ is a similarity mapping, because if $x \leqslant x'$ then there exists $c \in T$ such that $x \oplus c = x'$, thus $a \oplus (x \oplus c) = a \oplus x'$, consequently $a \oplus x < a \oplus x'$. The function θ satisfies the condition

$$(3.7) \qquad \forall_{a \in T} \ \{[\theta(a) = \alpha] \Rightarrow [\alpha(0) = a]\}$$

by (3.5). Moreover, if $\alpha, \beta \in \mathcal{T}$ and $\alpha(0) = a$, $\beta(0) = b$, then for any $t \in T$, $\alpha(t) = a \oplus t$ and $\beta(t) = b \oplus t$, so that $(\alpha \circ \beta)(t) = \alpha(b \oplus t) = a \oplus (b \oplus t) = (a \oplus b) \oplus t$ and hence

$$(3.8) \qquad \forall_{\alpha, \beta \in \mathcal{T}} \ \forall_{t \in T} \ (\alpha \circ \beta)(t) = (a \oplus b) \oplus t.$$

Thus, if $\alpha, \beta \in \mathcal{T}$, then $\alpha \circ \beta \in \mathcal{T}$. Let $<$ be an ordering relation in $\mathcal{T}$ such that

$$(3.9)\qquad \underset{\alpha,\beta\in\mathcal{T}}{\forall}\ \{(\alpha<\beta)\leftrightarrow[(a<b)\wedge(\alpha(0)=a)\wedge(\alpha(0)=b)]\}.$$

The monoid $(T, \oplus, <)$ is isomorphic to the monoid of movements $(\mathcal{T}, \circ, \leqslant)$ because by (3.8) we have

$$(3.10)\qquad \underset{a,b\in T}{\forall}\ \theta(a\oplus b)=\theta(a)\circ\theta(b),$$

and since for any $a,b,t\in T$, we have $a\leqslant b$ iff $a\oplus t\leqslant$ $\leqslant b\oplus t$, we obtain

$$(3.11)\qquad \underset{a,b\in T}{\forall}\ [(a\leqslant b)\leftrightarrow\theta(a)\leqslant\theta(b)].$$

Hence, if a monoid $(T, \oplus)$ satisfies conditions (3.1), (3.2) and (3.3), then $(T, \oplus)$ is a time generating. If a semigroup $(T, \oplus)$ is a time generating, then conditions (3.1), (3.2) and (3.3) are satisfied by Theorem 3.1.

4. Examples of general time sets

I. Let N be the set of all nonnegative integers, let $\leqslant$ be the natural ordering of N and let T be the ordinary addition of integers. The semigroup $(N,+)$ is a generator of the general time set $(N,\leqslant,\theta)$, where for any $k\in N$, $\theta(k)=$ $=\alpha_k$ and for any $n\in N$, $\alpha_k(n)=k+n$.

II. Let R_+ be the set of all nonnegative real numbers, let $\leqslant$ be natural ordering of R_+ and let + be the ordinary addition of real numbers. The semigroup $(R_+,+)$ is a generator of the general time set $(R_+,\leqslant,\theta)$, where for any $a\in R_+$, $\theta(a)=\alpha_a$ and for any $x\in R_+$, $\alpha_a(x)=a+x$.

III. Let N be the set of all nonnegative integers and let for any $l\in N$, $(N^{(l)},\leqslant)$ be an ordered set similar to $(N,\leqslant)$. We write $N^{(l)}=\{x : x=n^{(l)}\wedge n\in N\}$ and $n^{(l)}\leqslant m^{(l)}$ iff $n\leqslant m$. Let $T=\bigcup_{l\in N} N^{(l)}$ and let if $l<k$, then $n^{(l)}\leqslant m^{(k)}$ for any $n,m\in N$. The ordered set $(T,\leqslant)$ is not

similar to $(N, \leqslant)$. Let for any $n^{(1)} \in T$, $\theta(n^{(1)})$ be α such that for any $m^{(k)} \in T$,

$$(4.1) \qquad \alpha(m^{(k)}) = \begin{cases} (n+m)^{(1)}, & \text{for} \quad k = 0 \\ m^{(1+k)}, & \text{for} \quad k > 0. \end{cases}$$

It can be shown, that θ is a function of movements in $(T, \leqslant)$. If for any $n,m,l,k \in N$, $n^{(1)} \oplus m^{(o)} = (n+m)^{(1)}$ and $n^{(1)} \oplus m^{(k)} = m^{(1+k)}$ for any $k > 0$, then $(T, \oplus)$ is a generator of the general time set $(T, \leqslant, \theta)$. The semigroup $(T, \oplus)$ is not commutative, because $2^{(1)} \oplus 1^{(0)} = 3^{(1)}$ and $1^{(0)} \oplus 2^{(1)} = 2^{(1)}$.

IV. Let R_+ be the set of all nonnegative real numbers and let $\leqslant$ be natural ordering of R_+. Let for any $a \in R_+$, $\theta'(a)$ be α such that for any $t \in R_+$

$$(4.2) \qquad \alpha(t) = \begin{cases} (1-\varepsilon_a)\cdot t + a, & \text{for} \quad 0 \leqslant t < 1 \\ t+E(a) & \text{for} \quad t \geqslant 1, \end{cases}$$

where $\varepsilon_a = a - E(a)$ and $E(a)$ is an entier of a.

One can show, that θ' is a function of movements in $(R_+, \leqslant)$. If for any $a,b \in R_+$

$$(4.3) \qquad b \oplus a = \begin{cases} b+a-a\cdot\varepsilon_b, & \text{for} \quad 0 \leqslant a < 1 \\ b+a-\varepsilon_b, & \text{for} \quad a \geqslant 1 \end{cases}$$

then $(R_+, \oplus)$ is a generator of the general time set $(R_+, \leqslant, \theta')$. The semigroup $(R_+, +)$ and $(R_+, \oplus)$ are not isomorphic, because $\oplus$ is not commutative.

The examples III and IV would not be possible if a time set were defined similarly as in Definition 1.4.1 in [2], because then by Theorem 1.4.2 in [2] a commutativity condition would be satisfied.

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