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ON GROUPS HAVING THE PROPERTY W

In [1] we investigated subsets defined as follows

(1) $$K_w = \{g \in G : o(g) = w\},$$

where G is an arbitrary group in multiplicative notation, o(g) denotes the order of the element g, possibly infinity.

In the note mentioned above the following hypothesis is put forward: the set K_wK_w (subset multiplication) is an invariant subgroup of the group G. In [1] this hypothesis has been proved to be true in several particular cases.

P.Hall in [2], th.1, has constructed a locally finite group C with the following properties

(i) every finite group can be embedded in C,

(ii) any two isomorphic finite subgroups of C are conjugate in C,

(iii) the elements of C of the same order form one class S_m of elements conjugate in C and $S_mS_m = C$ for all $m > 1$,

(iv) the group C contains a continuum of distinct subgroups isomorphic to an arbitrary given countable locally finite group and a countable set of subgroups isomorphic to any given finite group.

Condition (iii) implies that C is a simple group. From (iii) it follows that $K_wK_w \leqslant C$ for each w expressing the order of an element in C. In [2], p.309 P.Hall adds the following remark: it would be interesting to know whether there exists a finite simple group with the property (iii). He suggests that the existence of such a group is rather dubious.

Properties of groups similar to (iii) have been also investigated by J.L.Brenner, M.Randall and J.Ridell in [6]. In that paper the authors showed that if G is a finite simple group and C- a class of conjugate non-identity elements of C, then there exists a smallest natural number $\nu = \nu(C)$ such that $C^{\nu} = CC\ldots C$. The connection between $\nu(C)$ and the order of elements of C was also investigated. It was shown that: 1) if $n > 6$, then in the alternation group there is no class C consisting of involutions such that $CC = A_n$; 2) if C-a class in A_n consisting of cycles of lenght 3, then $\nu(C) = [n/2]$. Also it is shown that if K is an infinite field then in the simple group PSL (n,K) there exists a class such that $\nu(C) = \frac{n^2-2}{2n-1}$.

In connection with the property 1) let us add that if in the alternating group, K_w denotes the set of all elements of order $w \neq 1$, then for $n < 5$, $K_wK_w = A_n$. One can check directly that in the symmetric group $S_n (n < 5)$ we have $K_wK_w \Delta S_n$, where K_w - the set of all elements of order w in the group S_n.

S.K.Stein [4] has investigated the decomposition of a group G (not necessarily simple) into subsets A,B such that $G = AB$. This problem was also investigated by A.D.Sandos [5] and others.

The property (iii) of th.1 of Hall [2] is related to the property W to be considered in this paper.

In the first part of my paper I will present a proof of the hypothesis stated above for the case of abelian groups, finite and infinite. Also examples will be given showing that there are non-abelian groups finite and infinite without the property W. The last part of the paper is devoted to description of non-abelian groups for which the hypothesis in question is valid.

D e f i n i t i o n 1. Let Ω denote the set of orders of the elements of a group G. We say that the group has property W if for every $w_i \in \Omega$ (where w_i may be ∞) $K_{w_i} K_{w_i}$ is a subgroup of the group G.

T h e o r e m 1. If the group G has property W, then the subgroup $K_w K_w$ is normal in the group G.

P r o o f . Let $b \in g K_w K_w g^{-1}$. Then $b = g a_1 a_2 g^{-1}$, where $a_1, a_2 \in K_w$. This implies $b = (g a_1 g^{-1})(g a_2 g^{-1}) = \bar{a}_1 \bar{a}_2$, where $\bar{a}_1, \bar{a}_2 \in K_w$, because $o(\bar{a}_1) = o(g a_1 g^{-1}) = w$, $o(\bar{a}_2) = o(g a_2 g^{-1}) = w$. Hence $b \in K_w K_w$, and $g K_w K_w g^{-1} \subset K_w K_w$ for all $g \in G$, which means that $K_w K_w$ is an invariant subgroup of the group G. This is also a characteristic subgroup, because isomorphism preserves property W.

Moreover let us observe that if $g \in K_w K_w$, then $g = g_1 g_2$, where $g_1, g_2 \in K_w$. Since $g^{-1} = g_2^{-1} g_1^{-1}$, $o(g_i) = o(g_i^{-1})$, we infer that $g_2^{-1}, g_1^{-1} \in K_w$, i.e. g^{-1} $K_w K_w$. Hence the proof of the fact that G has the property W can be reduced to showing that

$$K_w K_w K_w K_w \subset K_w K_w \tag{2}$$

(the set $K_w K_w$ is closed under group operation).

In the proof that abelian groups have property W we shall use the following theorem.

T h e o r e m 2. If groups $A_1, A_2, \dots, A_n$ have property W and $(o(A_i), o(A_j)) \neq 1$ for $i \neq j$, then the group $A_1 x A_2 x \dots x A_n$ has property W.

P r o o f . To simplify the notation we shall carry out the proof for $n = 2$ only, further generalizations is obvious. If $(w, o(A_1)) = 1$ and $w | o(A_2)$, then $K_w \subset A_2$, and if $(w, o(A_2)) = 1$ and $w | o(A_1)$, the thesis follows. If $(w, o(A_1)) = w_1 \neq 1$, $(w, o(A_2)) = w_2 \neq 1$, $w = w_1 w_2$, then by assumption we have

$$o(a_1, b_1) = o(a_2, b_2) \longrightarrow o(a_1) = o(a_2), o(b_1) = o(b_2),$$

and further

$$K_w = \left\{ (a_i,b_i) \in A_1 x A_2 : a_i \in K_{w_1} \subset A_1, b_i \in K_{w_2} \in A_2 \right\}.$$

By the definition of multiplication in the group $A_1 x A_2$ we have

(3) $$(a_1,b_1)(a_2,b_2)(a_3,b_3)(a_4,b_4) = (a_1a_2a_3a_4, b_1b_2b_3b_4).$$

By assumption, we have

$$\exists_{\bar{a}_1,\bar{a}_2 \in K_{w_1}} \quad a_1a_2a_3a_4 = \bar{a}_1\bar{a}_2$$

$$\exists_{\bar{b}_1,\bar{b}_2 \in K_{w_2}} \quad b_1b_2b_3b_4 = \bar{b}_1\bar{b}_2.$$

The element (3) can be written in the form

$$(\bar{a}_1\bar{a}_2, \bar{b}_1\bar{b}_2) = (\bar{a}_1,\bar{b}_1)(\bar{a}_2,\bar{b}_2),$$

where $(\bar{a}_i,\bar{b}_i) \in K_w$ $(i=1,2)$. This proves the inclusion (2).

Analogously one can prove the following theorem.

Theorem 3. If $A_1,\ldots,A_n$ have property W and the orders of the elements of the group A_i are relatively prime to the orders of all elements of the groups $A_j (i \neq j)$, then the group $A_1 x A_2 x \ldots x A_n$ has property W.

Theorem 4. If G is an abelian p-group, then G has property W.

Proof. The operation is performable in $K_w K_w$ if the following equality holds:

$$\forall_{g_1,g_2,g_3,g_4 \in K_w} \quad \exists_{a,b \in K_w} \quad g_1g_2g_3g_4 = ab,$$

where $w = p^m$.

Suppose that this equality did not hold. Then the product of every triple among the elements g_1,g_2,g_3,g_4 would have the order less than w. As well, combining elements of the product $g_1g_2g_3g_4$ in pairs we would obtain that the order of at least one pair is less that w. Among others, we would have $o(g_1g_2g_3) = p^{m_1} < w$, $o(g_2g_3g_4) = p^{m_2} < w$ and $o(g_1g_2) < w$ or $o(g_3g_4) < w$. Let $o(g_1g_2) = p^{m_3} < w$.

This implies $o((g_1g_2)^{-1}) = p^{m_3}$ and $o(g_3) = o((g_1g_2)^{-1}g_1g_2g_3) \leq \max(p^{m_1},p^{m_3}) < w$, which contradicts the assumption $o(g_3) = w$. If $o(g_1g_2) = w$, then making use of the elements g_3g_4 and $g_2g_3g_4$ we would obtain again a contradiction.

D e f i n i t i o n 2. We say that a group G has property W_1 if

$$\forall_{g_1,g_2,g_3 \in K_w} (g_1g_2g_3 = \bar{a}, \bar{a} \in K_w \vee g_1g_2g_3 = ab, a,b \in K_w.$$

R e m a r k . A group with property W need not possess property W_1. The example is provided by the group C_6.

T h e o r e m 5. Every abelian p-group has property W_1.

P r o o f . We consider an element of the form $a = a_1a_2a_3$, where $a_1,a_2,a_3 \in K_w$. Two cases are possible:

I. $a \in K_w$,

II. $o(a) = p^{k_1} < w = p^k (k_1 < k)$.

In the first case the theorem holds. In the second case $a = (a_1a_2)a_3$ and $o(a_1a_2) = p^k$. If we had $o(a_1a_2) = p^{k_2} < p^k$, then $o(a_3) = o((a_1a_2)^{-1}a)$ and $o(a_3) \leq [p^{k_2},p^{k_1}]$, whereas $o(a_3) = p^k > [p^{k_1},p^{k_2}]$.

T h e o r e m 6. Every torsion abelian group has property W.

P r o o f . We know that a torsion abelian group G is the direct product of its Sylov subgroups $S(p_i)$ (where p_i

are different primes and the number of subgroup may be infinity). However, for a fixed w all elements K_w of the group G belong to one subgroup $A = A_1 x A_2 x \ldots x A_n$ being the product of a finite number of Sylov subgroups corresponding to different primes $p_1, p_2, \ldots, p_n$. The validity of the thesis now follows from Theorem 3.

T h e o r e m 7. Every infinite abelian group G has property W.

P r o o f . The elements of finite order form a subgroup G_1 of G, which by Theorem 6 has property W. The remaining elements form one complex K_∞. Hence we have to prove that in $K_\infty K_\infty$ the operation is performable. Let $g_1, g_2, g_3, g_4 \in K_\infty$ and $g_1 g_2 g_2 g_3 = g$. To obtain the desired decomposition of g into a product of two elements of infinite order we apply the law of associativity to the product $g_1 g_2 g_3 g_4$. If this were not possible then we would obtain $o(g_1 g_2 g_2) = w_1 < \infty$, $o(g_2 g_3 g_4) = w_2 < \infty$ and the order of at least one of the pairs $g_1 g_2$ and $g_3 g_4$ would be finite. Let $o(g_1 g_2) = w_3 < \infty$, then $o((g_1 g_2)^{-1}) = w_3$ and $g_3 = (g_1 g_2)^{-1} g_1 g_2 g_3$, which implies $o(g_3) = o((g_1 g_2)^{-1} g_1 g_2 g_3) < [w_1, w_2]$, whereas $o(g_3) = \infty$.

The results obtained so far can be gathered in the following theorem.

T h e o r e m 8. Every abelian group has property W.

Now we shall show that there are groups without property W. An example of a group G in which $K_w K_w$ is not a subgroup of the group G for $w \neq 1$ is provided by the modular group of all transformations of the complex plane of the form

$$z' = \frac{az+b}{cz+d} \quad \text{where} \quad a,b,c,d \in Z,\ ad-bc = 1.$$

It can be shown ([9] p.211), that this group is the free product of two finite cyclic groups, one of which has order 2, the other has order 3. Also it can be shown ([9] p.209) that this group is isomorphic to the quotient group of matrices with integral entries, of order 2, with determinant 1, modulo its normal divisor

$$N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

i.e. PSL(2,Z). We can assume that PSL(2,Z) is generated by

$$t = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

We then have

$$t^2 = \begin{bmatrix} -1, & 0 \\ 0 & -1 \end{bmatrix} \in N, \quad s^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in N.$$

In the group PSL(2,Z) there are complexes K_1, K_2, K_3, K_∞. We shall show that $K_w K_w \not\leq PSL(2,Z)$ for $w = 2,3$. In fact, tN, sts^2N, tN, $ststs^2ts^3N \in K_2$, but the element which is their product

$$tsts^2tststs^2ts^2N$$

cannot be represented as a product of two elements belonging to K_2. This can be checked by a direct computation on matrices. Namely, we have

$$tsts^2tststs^2ts^2N = \begin{bmatrix} -7 & -3 \\ 12 & 5 \end{bmatrix} N.$$

The element

$$\begin{bmatrix} x & y \\ z & u \end{bmatrix} N$$

belongs to K_2 if and only if $z = -\frac{x^2+1}{y}$, $u = -x$. The assumption $y = 0$ gives the matrices belonging to N.
The matrix equation

$$\begin{bmatrix} -7 & -3 \\ 12 & 5 \end{bmatrix} N = \begin{bmatrix} x & y \\ -\frac{x^2+1}{y} & -x \end{bmatrix} N \begin{bmatrix} r & v \\ -\frac{r^2+1}{v} & -r \end{bmatrix} N$$

leads to the alternative of the following systems of equations

$$\begin{cases} \xi\left[xr - \frac{y}{v}(r^2+1)\right] = \eta(-7) \\ \xi(xv-yr) = \eta(-3) \\ \xi\left[-\frac{r}{y}(x^2+1) + \frac{x}{v}(r^2+1)\right] = \eta 12 \\ \xi\left[-\frac{v}{y}(x^2+1) + xr\right] = \eta 5 \end{cases}$$

where $\xi, \eta \in \{1,-1\}$.

Solving each of these equations we obtain the equation $(2v-r)^2+1 = 0$, which shows that the system has no solutions not only in integers, but also in real numbers.

For the elements of order 3 we have

$$tstN\ sN\ tstN\ sN = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} N.$$

Taking into account the fact that the elements

$$\begin{bmatrix} x & y \\ z & u \end{bmatrix} N \in K_3$$

have the form

$$\begin{bmatrix} x & y \\ -\frac{1}{y}(x^2+x+1) & -(x+1) \end{bmatrix} N$$

(the supposition $y = 0$ leads to a contradiction), we can check as previously that the matrix equation

$$\begin{bmatrix} x & y \\ -\frac{1}{y}(x^2+x+1) & -(x+1) \end{bmatrix} N \begin{bmatrix} u & v \\ -\frac{1}{v}(u^2+u+1) & -(u+1) \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} N$$

has no solution in integers.

There exist finite groups without property W. In fact, let $B = \{1,2\}$, $A = \{1,a,a^2,a^3\}$. By fun(B,A) we denote the set of all functions from B into A, i.e.

$$f_1 : \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \ f_2 : \begin{pmatrix} 1 & 2 \\ 1 & a \end{pmatrix}, \ f_3 : \begin{pmatrix} 1 & 2 \\ 1 & a^2 \end{pmatrix}, \ f_4 : \begin{pmatrix} 1 & 2 \\ 1 & a^3 \end{pmatrix},$$

$$f_5 : \begin{pmatrix} 1 & 2 \\ a & a^2 \end{pmatrix}, \ f_6 : \begin{pmatrix} 1 & 2 \\ a & a^3 \end{pmatrix}, \ f_7 : \begin{pmatrix} 1 & 2 \\ a^2 & a^3 \end{pmatrix}, \ f_8 : \begin{pmatrix} 1 & 2 \\ a & a \end{pmatrix},$$

$$f_9 : \begin{pmatrix} 1 & 2 \\ a^2 & a^2 \end{pmatrix}, \ f_{10} : \begin{pmatrix} 1 & 2 \\ a^3 & a^3 \end{pmatrix}, \ f_{11} : \begin{pmatrix} 1 & 2 \\ a & 1 \end{pmatrix}, \ f_{12} : \begin{pmatrix} 1 & 2 \\ a^2 & 1 \end{pmatrix},$$

$$f_{13} : \begin{pmatrix} 1 & 2 \\ a^3 & 1 \end{pmatrix}, \ f_{14} : \begin{pmatrix} 1 & 2 \\ a^2 & a \end{pmatrix}, \ f_{15} : \begin{pmatrix} 1 & 2 \\ a^2 & a \end{pmatrix}, \ f_{16} : \begin{pmatrix} 1 & 2 \\ a^3 & a^2 \end{pmatrix}.$$

Let us consider the mapping $\varphi : B \to$ fun (B,A) defined by the formula

$$\varphi(b) = f^b(x) = f(bx), \ x \in B.$$

By means of φ every element of the group B induces some automorphism of the group fun(B,A) and B is isomorphic with a subgroup of the group of automorphism of fun(B,A). We build a semi-direct product $B \lambda$ fun(B,A) with multiplication

$$(b_1f_i)(b_2f_j) = b_1b_2f_i^{b_2}f_j.$$

We shall show that $K_2K_2 \not\triangleleft B\lambda \mathrm{fun}(B,A)$. It is easy to see that

$$K_2 = \{f_3, f_9, f_{12}, 2f_1, 2f_6, 2f_9, 2f_{15}\}$$

and further

$$K_2K_2 = \{f_1, f_3, f_6, f_9, f_{12}, f_{15}, 2f_1, 2f_3, 2f_6, 2f_9, 2f_{12}, 2f_{15}, 2f_8, 2f_{10}\}$$

out this is no subgroup as it has 14 elements.

Other complexes are the following ones

$K_1 = \{f_1\}$,
$K_4 = \{f_2, f_4, f_5, f_6, f_7, f_8, f_{11}, f_{13}, f_{14}, f_{15}, f_{16}, 2f_3, 2f_8, 2f_{10}, 2f_{12}, f_{10}\}$
$K_8 = \{2f_2, 2f_4, 2f_5, 2f_7, 2f_{11}, 2f_{13}, 2f_{14}, 2f_{16}\}$.

It is easy to verify that

$$K_8K_8 = \{f_1, f_3, f_6, f_8, f_9, f_{10}, f_{12}, f_{15}\}$$

is an invariant subgroup of the group $B\lambda \mathrm{fun}\ (B,A)$.

Hence we can describe groups in which there is at least one complex K_s $(s \neq 1)$ such that $K_sK_s \triangleleft G$.

Now we consider non-abelian groups with property W.

Theorem 9. If $G = B\lambda A$ is a semi-direct product of groups B and A with the following properties:

(1) the group A has property W,
(2) the group B is abelian and has property W_1,
(3) $o(b_ia_i) = w \neq 1 \Leftrightarrow b_i \neq 1,\ o(b_i) = w$ or $b_i = 1,\ o(a_i) = w$,

then the group G has property W.

P r o o f . In order to prove that the operation is performable in $K_{w_G} K_{w_G}$, $(w \neq 1)$ we consider the product of pairs in K_{w_G}.

$$(4) \quad (b_1a_1)(b_2a_2)(b_3a_3)(b_4a_4) = b_1b_2b_3b_4a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4.$$

The following cases are possible

a) $b_1, b_2, b_3, b_4 \in K_{w_B}$,

b) $b = 1, b_2, b_3, b_4 \in K_{w_B}$,

c) $b_1 = b_2 = 1, b_3, b_4 \in K_{w_B}$,

d) $b_1 = b_2 = b_3 = 1, b_4 \in K_{w_B}$,

e) $b_1 = b_2 = b_3 = b_4 = 1$.

Ad a). Since B has property W, there exist $\bar{b}_1\bar{b}_2 \in K_{w_B}$ such that

$$(4) = \bar{b}_1\bar{b}_2a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4 = (\bar{b}_1 1)(\bar{b}_2a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4),$$

where in view of condition (3) we have

$$(\bar{b}_1 1), (\bar{b}_2a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4) \in K_{w_G} ,$$

Ad b) From condition (2) it follows that $b_2b_3b_4 = \bar{b}_1 \in K_{w_B}$, or $b_2b_3b_4 = \bar{b}_1\bar{b}_2$ $(\bar{b}_1, \bar{b}_2 \in K_{w_B})$. This implies

$$(4) = b_1\bar{b}_1a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4 = (b_1a_1)(\bar{b}_1a_2^{b_3b_4} a_3^{b_4} a_4),$$

where in view of condition (3) we have

$$(\bar{b}_1a_2^{b_3b_4} a_3^{b_4} a_4) \in K_{w_G}.$$

On the other hand

$$(4) = b_1\bar{b}_1\bar{b}_2 a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4 = (\bar{b}_1 1)(\bar{b}_2 a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4),$$

where $(\bar{b}_1 1), (\bar{b}_2 a_1^{b_2b_3b_4} a_2^{b_3b_4} a_3^{b_4} a_4) \in K_{w_G}$ on the basis of condition (3).

Ad c) $(4) = (1, a_1)(1a_2)(b_3a_3)(b_4a_4) = (b_3 a_1^{b_3} a_2^{b_3})(b_4a_4)$,

where $(b_3 a_1^{b_3} a_2^{b_3} a_3) \in K_{w_G}$ on the basis of condition (3).

Ad d) $(4) = (1\ a_1)(1\ a_2)(1\ a_3)(b_4a_4) = (1\ a_1)(b_4 a_2^{b_4} a_3^{b_4} a_4)$

and $(b_4 a_2^{b_4} a_3^{b_4} a_4) \in K_{w_G}$ by condition (3).

Ad e) In this case the validity of the thesis follows from condition 1 and next from condition 3.

In the sequel we shall consider semi-direct products $B \lambda A$ treating B as the group of automorphism of A.

In some cases the elements of B exhaust the set of non-zero endomorphisms of this group: $B = \text{End}\ A - \{0\}$. Since then all elements of B have inverses, $B \cup \{0\}$ is a field. Hence condition (3) of Theorem 9 can be formulated as follows. Let $(b_i a_i) \in K_{w_G}$ and $b_i \neq 1$. We then have

$$(b_i a_i)^w = b_i^w a_i^{b_i^{w-1}} a_i^{b_i^{w-2}} \dots a_i^{b_i} a_i = 1.$$

Consequently, if $b_i \neq 1$ condition (3) can be reduced to the following

$$(5) \qquad a_i^{b_i^{w-1}+b_i^{w-2}+\dots+b_i+1} = 1, \text{ if } B \cup \{0\} \text{ is a ring}$$

generated by the elements of the group B.

$$a^{\frac{1-b_i^w}{1-b_i}} = 1 \tag{6}$$

if $B \cup \{0\}$ is a field.

R e m a r k . A condition for the set of automorphisms of an abelian group to be a field is provided by a theorem of Schur ([7], p.263).

C o r o l l a r y 1. Let $G = B \lambda A$ be the semi-direct product of groups B and A satisfying the following conditions

a) A is an abelian group

b) B is the group of automorphisms of A, has property W_1 and upon adjoining 0 becomes a field. Then the group G has property W.

P r o o f . It suffices to show that condition (3) of Theorem 9 holds. This is guaranteed by the equality

$$a_i^{\frac{1-b_j^w}{1-b_j}} = \left(a_i^{1-b_j^w}\right)^{\frac{1}{1-b_j}} = (a_i^o)^{(1-b_j)^{-1}},$$

where $(1-b_j)^{-1}$ exists in a field for $b_j \neq 1$. If $b_j = 1$, the validity of the corollary results from the assumptions about the group A.

C o r o l l a r y 2. The group Aut $Z(p) \lambda Z(p)$, where p is a prime such that $p-1$ is a power of a prime $p-1 = p_1^n$ (where $p_1 = 2$ if $p > 2$) has property W.

In fact, Z(p) being an abelian group has property W. It is known that End $Z(p) \simeq Z(p)$, Aut $Z(p) = (\text{End } Z(p))^*$, hence Aut $Z(p) \simeq Z(p)^*$ and Aut $Z(p) \cup \{0\} \simeq Z(p)$, i.e. Aut Z(p) is an abelian p_1-group and consequently has property W_1. Since Aut $Z(p) \cup \{0\}$ is a field, it satisfies condition (3) of Theorem 9.

C o r o l l a r y 3. The group of a regular polygon of n angles has property W. In fact, it is known that this group is defined by the relations

$$a^n = 1,\ b^2 = 1,\ ba = a^{-1}b.$$

It is easy to show that this group is isomorphic to the semi-direct product $A \lambda Z(n)$, where A consists of two automorphisms: identity and $x \to -x$ of the group $Z(n)$. It is not difficult to check that condition 2 holds, the remaining conditions of Theorem 9 are also easily to verify.

Similarly it can be shown that the group $G = (a,b)$, where $b^2 = 1$, $(ab)^2 = 1$ has property W.

Corollary 4. A group of order pq (p,q - prime numbers) has property W.

Proof. We may assume that $p < q$. Groups of order pq can be described as follows ([3], p.63):

1) cyclic groups $a^{pq} = 1$,

2) non-abelian groups defined by the relations $a^p = 1$, $b^q = 1$, $ba = ab^r$, $r^p = 1 \pmod q$, $r \neq 1 \pmod q$, $p | q-1$.

In case 1) the validity of the thesis follows from the fact that G is abelian. In case 2) it is easy to show that G is a semi-direct product of two groups A ($a^p = 1$) and B ($b^q = 1$). The group B is a normal divisor, and A is a group of operators. From Theorem 5 it follows that A has property W_1. Hence it suffices to show that condition (3) of Theorem 9. Let $a^k \neq 1$, $o(a^k) = w$. Taking into account the relations describing operations in this group and the fact that $Z(q)$ is a field we have

$$(a^k b^i)^w = (a^k)^w (b^i)^{r^{(w-1)k} + r^{(w-2)k} + \ldots + r^k + 1} = a^{wk} \left(b^{r^{kw}-1}\right)^{\frac{i}{r^k-1}}.$$

We have clearly $r^k \neq 1 \pmod q$ for $k \neq p$. If w is the order of the element a^k, then $kw = sp$. By assumption $r^{wk} - 1 = (r^p)^s - 1 = cq$, i.e. $b^{r^{wk}-1} = (b^q)^c = 1$. Hence $o(a^k b^i) = w$, which shows that condition (3) of Theorem 9 holds.

Theorem 10. The group Aut $Q \lambda Q$ has property W.

P r o o f . The group Aut $Q\lambda Q$ is isomorphic to the group G of matrices of the form $\begin{bmatrix}1 & b\\ 0 & a\end{bmatrix}$, where $b \in Q$, $a \in Q^*$ ([8], p.66). Since

$$\begin{bmatrix}1 & b\\ 0 & a\end{bmatrix}^n = \begin{bmatrix}1 & b(1+a+\dots+a^{n-1})\\ 0 & a^n\end{bmatrix},$$

we see that the only complexes are K_1, K_2, K_∞. It is evident that $K_1K_1 \leqslant G$. We have

$$K_2 = \left\{\begin{bmatrix}1 & b\\ 0 & -1\end{bmatrix},\ b \in Q\right\}$$

and

$$\begin{bmatrix}1 & b_1\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & b_2\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & b_3\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & b_4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & b_1-b_2+b_3\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & b_4\\ 0 & -1\end{bmatrix},$$

which shows that the set K_2K_2 is closed under the group operation.

We have

$$K_\infty = \left\{\begin{bmatrix}1 & b\\ 0 & a\end{bmatrix},\ b \in Q, a \in Q^* \wedge (a \neq \pm 1 \vee a = 1,\ b \neq 0)\right\}.$$

Then

$$\begin{bmatrix}1 & b_1\\ 0 & a_1\end{bmatrix}\begin{bmatrix}1 & b_2\\ 0 & a_2\end{bmatrix}\begin{bmatrix}1 & b_3\\ 0 & a_3\end{bmatrix}\begin{bmatrix}1 & b_4\\ 0 & a_4\end{bmatrix} = \begin{bmatrix}1 & b\\ 0 & a_1a_2a_3a_4\end{bmatrix},$$

where $b = b_4+b_3a_4+b_2a_3a_4+b_1a_2a_3a_4$.

Since $a_1a_2a_3a_4 \neq 0$, there exist numbers $\bar{a}_1, \bar{a}_2, \neq \pm 1$ such that $\bar{a}_1\bar{a}_2 = a_1a_2a_3a_4$ and

$$\begin{bmatrix} 1 & b \\ 0 & a_1a_2a_3a_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{a}_1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & \bar{a}_2 \end{bmatrix}.$$

Hence in the set $K_\infty K_\infty$ the operation is also performable, which ends the proof.

T h e o r e m 11. The non-torsion group has property W.

P r o o f . In this group all elements without identity have infinite order. Hence there are only two complexes: K_1 and K_∞. Let $g_1, g_2, g_3, g_4 \in K_\infty$ and $g = g_1g_2g_3g_4$. The following cases are possible.

a) $g = 1$,

b) $o(g) = \infty$.

Ad a). The desired decomposition of the product is given by $g = (g_1g_2g_3)g_4$. In fact, $g_1g_2g_3 = g_4^{-1}$, hence $o(g_1g_2g_3) = o(g_4) = \infty$.

Ad b). Let $g = (g_1g_2)(g_3g_4)$. It is not possible that $o(g_1g_2) = 1$ and $o(g_3g_4) = 1$, because then $o(g) = 1$. Hence we have

$$o(g_1g_2) = \infty, \quad o(g_3g_4) = 1 \quad \text{and} \quad g = g_1g_2, \quad \text{or}$$

$$o(g_1g_2) = 1, \quad o(g_3g_4) = \infty, \quad \text{then } g = g_3g_4$$

or

$$o(g_1g_2) = \infty, \quad o(g_3g_4) = \quad , \quad \text{then} \quad g = (g_1g_2)(g_3g_4)$$

and the theorem holds.

T h e o r e m 12. If G is a finite nilpotent group and each of its Sylov subgroup has property W, then the group G has property W.

P r o o f . Making use of the theorem ([3] p.176) stating that a nilpotent group is decomposable into a direct product of its Sylov subgroups we have

$$G = G_{p_1^{\alpha_1}} \times G_{p_2^{\alpha_2}} \times \dots \times G_{p_n^{\alpha_n}}, \quad (\alpha_i \in N). \tag{7}$$

Since the orders of the subgroups appearing in (7) are pairwise prime, the validity of the theorem in question follows from Theorem 2.

T h e o r e m 13. The Hamilton group has property W.

P r o o f . The Hamilton group G_H can be decomposed into the direct product $G_1 \times G_2 \times G_3$, where G_1 - the group of quaternions, G_2 - an abelian group with finite elements of odd order, G_3 - an abelian group with elements of the second order. For the complex K_1 and the complexes of odd order the property W is obvious. G_H also has complexes of elements of the orders 2s and 4s, where s is an odd number expressing the order of an element in G_2. We consider the question whether the operation is performable in $K_{2s}K_{2s}$. Since G_1, G_2, G_3 are normal divisors in G_H and G_H is their direct product, we have

$$(k_1h_1m_1)(k_2h_2m_2)(k_3h_3m_3)(k_4h_4m_4) = \prod_1^4 k_i \prod_1^4 h_i \prod_1^4 m_i,$$

where h_i are elements of order s in the group G_2. Hence we infer that

$$\underset{h_5,h_6 \in K_{s_{G_2}}}{\exists} \prod_1^4 h_i = h_5h_6.$$

For the remaining products the following cases are possible:

1) $\prod_1^4 k_i = 1$, $\prod_1^4 m_i = 1$, but $a^2a^2 = 1$ and $\prod_1^4 (k_ih_im_i) =$

$= (a^2h_5 1)(a^2h_6 1)$,

2) $\prod_1^4 k_i = 1$, $\prod_1^4 m_i = m_5 \neq 1$, then $\prod_1^4 (k_ih_im_i) = (a^2h_5m_5)(a^2h_6 1)$,

3) $\prod_1^4 k_i = k_5 = a^2$, $\prod_1^4 m_i = 1 = m_5m_5^{-1}$, where m_5 is any element of order 2. Hence we have $\prod_1^4 (k_ih_im_i) = (a^2h_5m_5)(1h_6m_5^{-1})$,

4) $\prod_1^4 k_i = k_5 = a^2$, $\prod_1^4 m_i = m_5 \neq 1$, then $\prod_1^4 (k_i h_i m_i) = (a^2 h_5 1)(1 h_6 m_5)$.

Hence in all cases we obtain a product of two elements belonging to $K_{2s} \subset G_H$. The thesis also holds for complexes of order 4s, because the triple (k_i, h_i, m_i) belongs to the complex K_{4s} of the group G_H as k_i is an element of order 4 and h_i is of order s. Since for G_1, G_2 the property W holds (for G_1 this can be checked directly), we obtain

$$\prod_1^4 (k_i h_i m_i) = \prod_1^4 k_i \prod_1^4 h_i \prod_1^4 m_i = k_5 k_6 h_5 h_6 m_5 = (k_5 h_5 m_5)(k_6 h_6 1),$$

where $k_5, k_6 \in K_{4_{G_1}}$, $h_5, h_6 \in K_{s_{G_2}}$, m_5 - any element of G_3.

T h e o r e m 14. Groups of order p^3 (p-a prime number) have the property W.

P r o o f . Groups of order p^3 can be described as follows:

1) for p=2

a) the abelian group of order 8,

b) the group of quaternions,

c) the group of symmetries of a square.

In the cases above, the validity of the theorem follows:
a) from the fact that it is abelian, b) by a direct verification, c) by Corollary 3 of Theorem 9.

2) for $p \neq 2$

a) the abelian group of order p^3,

b) a non-abelian group such that $a^{p^2} = 1$, $b^p = 1$, $b^{-1}ab = a^{p+1}$,

c) a non-abelian group such that $a^p=1$, $b^p=1$, $c^p=1$, $ab=bac$, $ca=ac$, $cb=bc$.

In the case a) the theorem holds since the group is abelian.
Ad b). Let $A = \{1, a, \dots, a^{p^2-1}\}$, $B = \{1, b, \dots, b^{p-1}\}$, $G = BA$. G has the following complexes: K_1, K_p, K_{p^2}. We have

$$(b^k a^i)^p = b^{kp}(a^i)^{(p+1)^{(p-1)k}+(p+1)^{(p-2)k}+\ldots+(p+1)^k+1},$$

$$b^{kp}=1 \text{ and } a^{i((p+1)^{(p-1)k}+(p+1)^{(p-2)k}+\ldots+(p+1)^k+1)} = 1 \Longleftrightarrow$$

$$p^2 \mid i((p+1)^{(p-1)k}+(p+1)^{(p-2)k}+\ldots+(p+1)^k+1).$$

Making use of Newton s binomial formula, we obtain

$$(p+1)^{(p-1)k}+(p+1)^{(p-2)k}+\ldots+(p+1)^k+1=(p^{(p-1)k}+(p-1)kp^{(p-1)k-1}+\ldots$$

$$\ldots+(p-1)kp+1)+(p^{(p-2)k}+(p-2)kp^{(p-2)k-1}+\ldots+(p-2)kp+1)+ \ldots$$

$$\ldots+(p^k+kp^{k-1}+\ldots+kp+1)+1=W(p)p^2+((p-1)kp+(p-2)kp+ \ldots$$

$$\ldots+kp)+p=W(p)p^2+\frac{kp+(p-1)kp}{2}(p-1)+p=W(p)p^2+kp^2\,\frac{p-1}{2}+p=(\ )^*.$$

Since $2 \nmid p$, we have $2 \mid p-1$, i.e. $p \mid (\ )^*$, $p^2 \nmid (\ )^*$. Hence $(b^k a^i) \in K_{p^2_G} \Longleftrightarrow p \nmid i$.

It is clear that the operation is performable in K_{1_G} K_{1_G}.

Let us consider the product

$$(\ )^{**} = (b^{k_1}a^{i_1})(b^{k_2}a^{i_2})(b^{k_3}a^{i_3})(b^{k_4}a^{i_4})$$

of the elements $(b^{k_j}a^{i_j}) \in K_{p_G}$, such that

$$(b^{k_j})^p = 1,\ (a^{i_j})^p = 1,\ b^{k_j} \neq 1,\ a^{i_j} \neq 1.$$

Then we have

$$(\ )^{**} = b^{k_1+k_2+k_3+k_4} a^{i_1(p+1)^{k_2+k_3+k_4}+i_2(p+1)^{k_3+k_4}+i_3(p+1)^{k_4}+i_4} =$$

$$= \begin{cases} 1\ 1=(b^{k_6}1)((b^{k_6})^{-1}1), \\ 1\ a^{i_5}=(b^{k_6}1)((b^{k_6})^{-1}a^{i_5}), \\ b^{k_5}1=(b^{k_5}a^{i_6})(1(a^{i_6})^{-1}), \\ b^{k_5}a^{i_5}=(b^{k_5}1)(1\ a^{i_5}), \end{cases}$$

where b^{k_5}, a^{i_5} are elements of order p, which follows from the fact that A and B are abelian and b^{k_6}, a^{i_6} are arbitrarily chosen elements of order p.

Since $(b^k a^i) \in K_{p_G^2} \Leftrightarrow p \nmid i \leftrightarrow a^i \in K_{p_A^2}$, we see that the equality $(K_w)^4 = (K_w)^2$ is evident for complexes.

Ad c). Let $A = \{1,a,\dots,a^{p-1}\}$, $B = \{1,b,\dots,b^{p-1}\}$, $C = \{1,c,\dots,c^{p-1}\}$, $G = ABC$. G has only two complexes K_1 and K_p, where $(a^k b^i c^j) \in K_{p_G}$ with $a^k \neq 1$, or $b^i \neq 1$, or $c^j \neq 1$. Since $K_{p_G} K_{p_G} = G$, it is clear that property W holds.

T h e o r e m 15. The group of order p^n $(n > 3)$ defined by the relations $b^p = 1$, $a^{p^{n-1}} = 1$, $bab^{-1} = a^r$, $r = p^{n-2}+1$, $p \neq 2$ has property W.

P r o o f . Let $G = AB$, $B = \{1,b,\dots,b^{p-1}\}$, $A = \{1,a,\dots,a^{p^{n-1}-1}\}$. Let us find the order of the element $(b^k a^i) \in G$.

$$(b^k a^i)^{p^s} = b^{kp^s} a^{i(r^{(p^s-1)k}+r^{(p^s-2)k}+\dots+r^k+1)} = b^{kp^s} a^{iW(p)},$$

$$W(p)=(p^{n-2}+1)^{(p^s-1)k}+(p^{n-2}+1)^{(p^s-2)k}+\dots+(p^{n-2}+1)^{k}+1=$$

$$p^{(n-2)(p^s-1)k}+\dots+(p^s-1)kp^{n-2}+1)+(p^{(n-2)(p^s-2)k}+\dots$$

$$\dots+(p^s-2)p^{n-2}k+1)+\dots+(p^{(n-2)k}+\dots+kp^{n-2}+1)+1=p^{(n-2)(p^s-1)k}+\dots$$

$$+((p^s-1)kp^{n-2}+(p^s-2)kp^{n-2}+\dots+1\ kp^{n-2})+p^s=$$

$$=W_1(p)(p^{n-2})^2+kp^{n-2}p^s\frac{p^s-1}{2}+p^s.$$

Since $2\nmid p$, i.e. $2\nmid p^s$, $2\mid p^s-1$, $1<s<n-1$. This implies $p^s/W(p), p^{s+1}\nmid W(p)$, i.e. $p^{n-1}/iW(p)\Leftrightarrow p^{n-1-s}/i\Leftrightarrow o(a^i)<p^s$ and $(b^k)^{p^s}=(b^{p^s})^k=1$ for every $s\geqslant 1$. Thus the group G has the following complexes

$$K_1,K_p,\dots,K_{p^s},\dots,K_{p^{n-1}},$$

where

$$K_1=\{(1\ 1)\},K_p=\{(b^ka^i)\in G: o(b^k)\leqslant p, o(a^i)<p, \sim(b^k=1\wedge a^i=1)\}$$

and all K_{p^s} with $s>1$ are of the form

$$K_{p^s}=\{(b^ka^i)\in G: o(b^k)<p,\ o(a^i)=p^s\}.$$

Let $(b^ka^i)\ K_p$, then we have

$$(b^{k_1}a^{i_1})(b^{k_2}a^{i_2})(b^{k_3}a^{i_3})(b^{k_4}a^{i_4})=\begin{cases}1\ 1=(b^{k_6}1)((b^{k_6})^{-1}1),\\ 1a^{i_5}=(b^{k_6}1)((b^{k_6})^{-1}a^{i_5}),\\ b^{k_5}1=(b^{k_5}a^{i_6})(1(a^{i_6})^{-1}),\\ b^{k_5}a^{i_5}=(b^{k_5}1)(1\ a^{i_5}),\end{cases}$$

where b^{k_5}, a^{i_5} are elements of order p (since B and A are abelian) and b^{k_6}, a^{i_6} - are arbitrarily chosen elements of order p.

Let $(b^k a^i) \in K_{p^s}$, $s > 1$. We then have

$$( )^* = (b^{k_1}a^{i_1})(b^{k_2}a^{i_2})(b^{k_3}a^{i_3})(b^{k_4}a^{i_4}) =$$

$$= b^{k_1+k_2+k_3+k_4}\, a^{i_1 r^{k_2+k_3+k_4} + i_2 r^{k_3+k_4} + i_3 r^{k_4} + i_4} =$$

$$= b^{k_1+k_2+k_3+k_4} a^{i_1 r^{k_2+k_3+k_4}}\, a^{i_2 r^{k_3+k_4}}\, a^{i_3 4^{k_4}}\, a^{i_4},$$

where

$$o\left(a^{i_1 r^{k_2+k_3+k_4}}\right) = o(a^{i_1}),\quad o\left(a^{i_2 r^{k_3+k_4}}\right) = o(a^{i_2}),\quad o\left(a^{i_3 r^{k_4}}\right) = o(a^{i_3})$$

Since A is abelian, we finally obtain

$$( )^* = b^{k_1+k_2+k_3+k_4} a^{i_5} a^{i_6} = \left(b^{k_1+k_2+k_3+k_4} a^{i_5}\right)(1\, a^{i_6}),$$

where $o(a^{i_5}) = o(a^{i_6}) = p^s$.

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