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OPERATORS IN MULTI-VALUED LOGIC

1. Introduction

In [6] W.Szenajch defined a certain operator P in multi-valued logic and showed its functional completeness in three-valued logic. He also constructed a pneumatic technical element realizing this operator and using this element he built pneumatic logical systems in three-valued logic.

The aim of this paper is the investigation of the properties of operators in multi-valued logic, in particular the question of their functional completeness. In section 2 we recall some criteria for functional completeness of a set of functions with arguments and values in a finite set. In section 3 we give a general definition of the operator with n inputs and p outputs, and we prove that the operator P defined by W.Szenajch is functionally complete in multi-valued logic (theorem 1). We also define another operator R and we show that it is functionally complete in multi-valued logic (theorem 2). In section 4 we give a necessary and sufficient condition in order that an operator be functionally complete in multivalued logic (lemma 1) and we characterize the functional completeness of an operator with single input (lemma 2). Next we prove that the operator R defined in section 3 is the simplest one among functionally complete operators in k-valued logic ($k \geqslant 3$) in the sense that it depends upon a minimal number of variables.

2. Criteria for the functional completeness of a set of functions

Let A be a k-element set ($|A| = k$). We denote by $F_A^{(n)}$ the set of all n-argument functions with arguments and values in A and let $F_A = \bigcup_{n=1}^{\infty} F_A^{(n)}$. Let $G \subset F_A$. By $[G]$ we denote the set of all those $f \in F_A$ which are superpositions of functions belonging to G. We say that G is functionally complete in A if $[G] = F_A$. A function $g \in F_A$ such that $[\{g\}] = F_A$ is called a Sheffer function or a functionally complete function. Let $\varrho \subset A^m$ be an m-argument relation in A. We say that $f \in F_A^{(n)}$ preserves ϱ provided that for any $r_i = (r_i(1),\ldots,r_i(m)) \in \varrho$ with $i = 0,1,\ldots,n-1$, we have

$$(1)\quad (f(r_0(1),\ldots,r_{n-1}(1)),\ldots,f(r_0(m),\ldots,r_{n-1}(m))) \in \varrho .$$

There are known characterizations ([2] for $|A| = 3$, [4] for $|A| > 3$) of maximal closed subsets of F_A, i.e. subsets $G \subset F_A$ such that $G = [G] \neq F_A$, but $[G,\{f\}] = F_A$ for every function $f \notin G$. Namely, G is a maximal closed subset of F_A if and only if every function $g \in G$ preserves a relation in A belonging to one of distinguished types of relations in A. We will not list all those distinguished types of relations, let us mention only that if every function $h \in H \subset F_A$ preserves a relation belonging to one of the distinguished types (e.g. any non-trivial equivalence relation in A), then H is contained in some maximal closed subset G containing this relation and hence $[H] \subset [G] \neq F_A$, which means that H is not functionally complete in A.

Now let us assume that A is a Post algebra of the following form: $A = L_k = (0,1,\ldots,k-1,\{0,k-1\})$, where $k \geq 2$, with constants $0,1,\ldots,k-1$ and Boolean centre $B = \{0,k-1\}$ under the operations:

$$v_1 \vee v_2 = \max(v_1,v_2)$$

$$v_1 \wedge v_2 = \min(v_1,v_2).$$

Every element $v \in L_k$ can be represented in the form

$$v = \bigcup_{i=0}^{k-1} C_i(v) \wedge i,$$

where $C_i(v) = \begin{cases} 0 \text{ for } v \neq i \\ k-1 \text{ for } v = i \end{cases}$ are Boolean coefficients in the so-called disjoint representation of an element of a Post algebra [7]. The algebra L_k is called the algebra of k-valued logic. It is known that every function $f \in F_{L_k}^{(n)}$ can be represented in the form

$$(2) \quad f(v_1,\dots,v_n) = \bigcup_{j=0}^{k^n-1} C_{a_1^j}(v_1) \wedge \dots \wedge C_{a_n^j}(v_n) \wedge f(a_1^j,\dots a_n^j),$$

where $a_1^j,\dots,a_n^j$ with variable j are all n-term sequences with elements in L_k.

From (2) it follows that all constant functions, two binary functions ($\vee$ and $\wedge$) and k one argument function $C_i(v)$, where $i = 0,1,\dots k-1$, form a functionally complete set of functions in L_k. Several functionally complete sets of functions in L_k are known. E.g. J.Słupecki [5] in 1939 gave the following criterion. Let $G \subset F_{L_k}$ and let $F_{L_k}^{(1)} \subset G$. Then $[G] = F_{L_k} \Leftrightarrow G$ contains at least one irreducible 2-argument function whose range equals to L_k. Let us recall that an n-argument function is said to be irreducible if it essentially depends upon n-variables. In [1] a similar criterion has been proved with the modification that the set G contains any irreducible n-argument function.

3. <u>The operators $P(z_0,z_1,z_2,v_1,v_2)$ and $R(z_0,z_1,v_1,v_2)$</u>

We say that a family $\{V_i\}_{i=0,1,\dots,p-1}$ of subsets of the set $V = L_k^n$ is a partition of V if $V_i \cap V_j \neq \emptyset$ for $i \neq j$ and $\bigcup_{i=0}^{p-1} V_i = V$. Let P denote such a partition and

$p = |P|$ the number of elements of this family. Let us define a function O: $L_k^p \times L_k^n \longrightarrow L_k$ of $p+n$ variables as follows

(3) $O(z_0,\ldots,z_{p-1}, v_1,\ldots,v_n) = z_i$ for $(v_1,\ldots,v_n) \in V_i$,

where $i=0,1,\ldots,p-1$.

The pair $\langle P,O \rangle$ is called an operator in the logic L_k. If the partition P is fixed, we also call the function $O(z_0,\ldots,z_{p-1}, v_1,\ldots,v_n)$ an operator, and we call the variables $v_1,\ldots,v_n$ inputs and the variables $z_0,\ldots,z_{p-1}$ outputs of the operator. An example of an operator with two inputs and three outputs was provided by W.Szenajch [6]. Namely,

$$(4) \qquad P_k(z_0,z_1,z_2,v_1,v_2) = \begin{cases} z_0 & \text{for } v_1=v_2 \\ z_1 & \text{for } v_1 > v_2 \\ z_2 & \text{for } v_1 < v_2 . \end{cases}$$

In this case $n=2$, and $P = \left\{V_i : V_i \subset L_k^2,\ i=0,1,2\right\}$ where

$$V_0 = \left\{(v_1,v_2) \in L_k^2 : v_1 = v_2\right\},$$
$$V_1 = \left\{(v_1,v_2) \in L_k^2 : v_1 > v_2\right\},$$
$$V_2 = \left\{(v_1,v_2) \in L_k^2 : v_1 < v_2\right\}.$$

In [6] W.Szenajch deals with the synthesis of three valued pneumatical logical elements. Input signals can take on three values distinguished by the level of pressure

0 ———— 0 kG/cm^2

1 ———— 2-2,5 kG/cm^2

2 ———— 4-6 kG/cm^2.

Although there are known pneumatic elements realizing the functions $\vee$, $\wedge$, and $C_i(v)$, $i = 0,1,2$ in three-valued logic, by means of which it is possible to express any function belonging to F_{L_k}, it turned out that logical systems built with the aid of these functions are very complicated. The operator P_k introduced by W.Szenajch is more suitable for this aim. W. Szenajch has proved that this operator is functionally complete for $k = 3$.

In the definition of the operator P_k it is not essential that input or output signals are k-valued. This fact is not essential when we construct technical elements realizing the operator P_k. Namely, it suffices to differentiate the levels of signals so that they correspond to the signals $0,1,\ldots,k-1$ and the technical element corresponding to P_3 can work in k-valued logic. Hence we can omit the index k in the symbol P_k and we may denote it by P. The question of the functional completeness of the operator P in an arbitrary logic is decided in the following theorem.

T h e o r e m 1. The operator $P(z_0,z_1,z_2,v_1,v_2)$ is functionally complete in L_k for any k.

P r o o f . It is easy to verify that

$$i = P(i,i,i,v_1,v_2) \quad \text{for } i = 0, 1,\ldots,k-1,$$

$$v_1 \vee v_2 = P(v_1,v_1,v_2,v_1,v_2),$$

$$v_1 \wedge v_2 = P(v_1,v_2,v_1,v_2),$$

$$C_i(v) = P(k-1,0,0,v,i) \quad \text{for } i = 0,1,\ldots,k-1,$$

and hence by (2) we infer that any function belonging to F_{L_k} can be realized by means of the operator P.

From Theorem 1 it follows that with the aid of the existing technical elements realizing the operator P_3 we can build systems in k-valued logic for $k > 3$. However, it seems that Szenajch's operator is suitable mostly for 3-valued logic. In fact, let $f \in F^{(2)}_{L_3}$ and let $f(i,j) = z^{ij}$ for $i,j = 0,1,2$. It is easy to see that

$$(5) \quad f(v_1,v_2) = P\left[P(z^{11},z^{12},z^{10},v_2,1),P(z^{21},z^{22},z^{20},v_2,1),\right.$$
$$\left. P(z^{01},z^{02},z^{00},v_2,1),v_1,1\right].$$

The function f can be expressed by means of the operator P, because the comparison of each variable with 1 determines the value of the variable. In the case of four-valued logic, this method is not valid. Namely, let $g \in F_{L_4}$ where $g(i,j) = z^{ij}$ for $i,j=0,1,2,3$. It is easy to verify that

$$
\begin{aligned}
(6) \qquad g(v_1,v_2) = P\Big\{ & P\big[z^{11},P(z^{12},z^{13},-,v_2,2),z^{10},v_2,1\big], \\
& P\big[P(z^{21},P(z^{22},z^{23},-,v_2,2),z^{20},v_2,1), \\
& P(z^{31},P(z^{32},z^{33},-,v_2,2),z^{30},v_2,1),-,v_1,2\big], \\
& P\big[z^{01},P(z^{02},z^{03},-,v_2,2),z^{00},v_2,1\big],v_1,1\Big\}.
\end{aligned}
$$

In the formula above, the inputs marked with a bar are redudant. The form (6) of a function of two variables is not as simple as (5), which is due to the fact that the variables in fourvalued logic have to be compared with two elements, say the elements 1 and 2, in order to determine their values. This justifies a hypothesis that in k-valued logic ($k > 3$) the operator P will not be as useful as in three-valued logic. Instead we can propose a simpler operator $R(z_0,z_1,v_1,v_2)$ defined as follows

$$
(7) \qquad R(z_0,z_1,v_1,v_2) = \begin{cases} z_0 & \text{for} \quad v_1 \geqslant v_2 \\ z_1 & \text{for} \quad v_1 < v_2. \end{cases}
$$

The construction of a technical element realizing this operator could be essentially simpler that that of the element P (fewer movable parts, only two inputs), but its technical virtues consisting in comparison of the level of pressure would be the same as previously.

T h e o r e m 2. The operator $R(z_0,z_1,v_1,v_2)$ is functionally complete in L_k for any k.

P r o o f . It is easy to verify that

$i = R(i,i,v_1,v_2)$ for $i = 0,1,\dots,k-1$,

$v_1 \vee v_2 = R(v_1,v_2,v_1,v_2)$,

$v_1 \wedge v_2 = R(v_2,v_1,v_1,v_2)$,

$C_i(v) = R(k-1,0,v,i) \wedge R(0,k-1,v,i+1)$ for $i=0,1,\dots,k-2$,

$C_{k-1}(v) = R(k-1,0,v,k-1)$,

which by (2) proves that R is functionally complete in L_k.

Let us observe that the operator P is a function of five variables, whereas the operator R is a function of four variables. There arises the question, whether there exists an operator with the number of variables less than 4, which is functionally complete in L_k for $k \geqslant 3$. In the next paragraph it will be shown, that the answer to this question is in the negative.

4. Functional completeness of operators in L_k

Functionally complete operators are some particular Sheffer's functions. It is known that there are Sheffer's functions of two variables only (e.g. a function of Webb [8], defined by $f(x_0,x_1) = ((|x_0 \vee x_1) \oplus 1) \pmod k$ is a Sheffer function in L_k), however, technical construction of logical systema by means of the function is rather complicated. In various domains of engineering one can apply operators of the form (3) which are more suitable for this aim. Examples of such operators are provided by the operators P and R, where the partitions of the set $V = L_k^2$ is compatible with technical properties of pneumatic elements.

Now we shall give several properties of functionally complete operators in L_k.

L e m m a 1. Every operator $\langle P,O \rangle$ that is functionally complete in L_k must possess at least two outputs, i.e. $p = |P| \geqslant 2$.

P r o o f . Let $p = 1$. Then the operator has the form $O(z_0,v_1,\dots,v_n) = z_0$ for every $(v_1,\dots,v_n) \in L_k^n$. This is so-called trivial operator. Since $O(O(z_0,v_1,\dots,v_n),v_1,\dots,v_n) = z_0$, $O(z_0,v_1,\dots,O(z_0,v_1,\dots,v_n),\dots,v_n) = z_0$, no irreducible function g of two variables belongs to $[\{0\}]$. Hence by Słupecki's criterion given in Chapter 2 the operator P is not functionally complete in L_k.

L e m m a 2. Let $\langle P,O \rangle$ be an operator with one input. Then $\langle P,O \rangle$ is functionally complete in L_k if and only if O is a function of $k+1$ variables, i.e. $p = |P| = k$.

P r o o f . If $p = |P| = k$, then the sets V_i forming a partition of L_k are one-element sets. Without loss of generality we may assume that $V_i = \{i\}$ for $i = 0,1,\dots k-1$. We then have $O(z_0,\dots,z_{k-1},v) = z_i$ for $v = i$, where $i = 0,1,\dots,k-1$. This is the so-called T-gate defined for three-valued logic in [1]. We shall denote this operator by T. Any function $f \in F^{(1)}_{L_k}$ can be represented in the following form

$$f(v) = T(f(0),f(1),\dots,f(k-1),v).$$

The function $g \in F^{(2)}_{L_k}$ defined as follows

$$g(v_1,v_2) = T\Big[T(0,1,\dots,k-1,v_1),T(1,2,\dots,k-1,0,v_1),\dots,$$
$$T(k-1,0,\dots,k-3,k-2,v_1),v_2\Big]$$

satisfies, as is easy to see, the conditions $g(0,v_2) = v_2$, $g(v_1,0) = v_1$, which show that g is irreducible and the set of values of g equals L_k. By Słupecki's criterion, we infer that T is functionally complete in L_k.

Now assume that $p = |P| \neq k$, which in view of $|L_k| = k$ means that $p < k$. Thus the operator is of the form $O(z_0,\dots,z_{p-1},v)$, where $p < k$. We consider the following cases:

1^o $k = 2$. Then $p = 1$ and by Lemma 1 O is not functionally complete.

2^o $k \geqslant 3$, $p = 1$. Then by Lemma 1 O is not functionally complete.

3^o $k > 3$ and $p > 1$, i.e. $1 < p < k$.
Then the partition $P = \{V_0, V_1, \ldots, V_{p-1}\}$ determines in L_k a nontrivial equivalence selection ϱ, whose equivalence classes are the sets V_i, where $i = 0,1,\ldots,p-1$. Let $r_i = (r_i(1), r_i(2)) \in \varrho$, where $i = 0,1,\ldots,p$ and $r_p(1)$, $r_p(2) \in V_{i_0}$, with $i_0 \in \{0,1,\ldots,p-1\}$. Then we have $(O(r_0(1),\ldots,r_{p-1}(1), r_p(1)), O(r_0(2),\ldots,r_{p-1}(2), r_p(2)) = (r_{i_0}(1), r_{i_0}(2)) \in \varrho$, which proves that O preserves the relation ϱ so that it is not functionally complete (see Chpt.2). Thus the lemma has been proved.

C o r o l l a r y . The T-gate is the only operator with one input which is functionally complete.

T h e o r e m 3. For even $k > 3$ we have

(a) there exists a functionally complete operator in L_k which is a function of four variables.

(b) every functionally complete operator in L_k is a function of at least four variables.

P r o o f . (a) The operator $R(z_0, z_1, v_1, v_2)$ being a function of four variables is functionally complete in L_k for every k, in particular for $k > 3$.

(b) Assume that an operator is a function of fewer than four variables. If it has one input, then it is not functionally complete by Lemma 2. If it has two inputs, then it is not functionally complete by Lemma 1.

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