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BASES OF AGGREGABLE SETS

Introduction

The notions of a computable set and a computable function are introduced in [1]. The notions of an aggregable set and a basis of an aggregable set are introduced in this paper analogously to [2]. The aim of this paper is to evolve a theory of bases of aggregable sets.

1. Aggregable sets

Let $\zeta = (T, \leqslant, \Theta, X, Y, Z)$ be such a system which is introduced in [1]. By $\mathcal{F}_\zeta$ we denote the set of all computable functions $f : T \times X \dot{\to} Y$.

Definition 1.1. Functions $f, g \in \mathcal{F}_\zeta$ are said to be commonly computable (in symbols $f \sim g$) if there exists a computable set $F \subseteq \mathcal{F}_\zeta$ such that $f, g \in F$.

Corollary 1.1. The relation $\sim$ is reflexive and symmetric in $\mathcal{F}_\zeta$.

Definition 1.2. The set $F \subseteq \mathcal{F}_\zeta$ is called aggregable iff the relation $\sim$ is transitive in F and the set F is $*$-closed (see [1]).

Corollary 1.2. Any computable set is aggregable.

Corollary 1.3. For any aggregable set F and for any set $H \subseteq F$ the set H^* (see [1]) is aggregable.

Definition 1.2 and Corollary 1.1 imply the following corollary.

Corollary 1.4. The relation $\sim$ is an equivalence relation in any aggregable set.

Theorem 1.1. For any aggregable set F and for any $f \in F$ the equivalence class $[f]_\sim$ is a computable set.

Proof. The proof is analogical to the proof of Theorem 2 in [2].

2. $*$-connected computable sets

Definition 2.1. For any $f,g \in \mathcal{F}_\zeta$ we define a relation $\overset{*}{=}$ as follows (see [1])

$$(f \overset{*}{=} g) \Leftrightarrow \underset{a,b\in T}{\exists} (f_a = f_b). \tag{2.1}$$

Theorem 2.1. The relation $\overset{*}{=}$ is an equivalence relation in $\mathcal{F}_\zeta$.

Proof. It is evident that the relation $\overset{*}{=}$ is reflexive and symmetric in $\mathcal{F}_\zeta$. If $f \overset{*}{=} g$ and $g \overset{*}{=} h$, then there exists $a,b,c,d \in T$ such that $f_a = g_b$ and $g_c = h_d$. Let $b \leqslant c$ and $\Theta(b) = \beta$, $\Theta(c) = \gamma$ (see [1]). Let $e = \beta^{-1}(c)$ and $\Theta(e) = \eta$. If $f_a = g_b$, then $(f_a)_e = (g_b)_e = g \circ \tilde{\beta} \circ \tilde{\eta} = g \circ \tilde{\gamma} = g_c$, thus $f_{a'} = g_c$, where $f_{a'} = (f_a)_e$. So, if $g_c = h_d$, then $f_{a'} = h_d$, whence $f \overset{*}{=} h$. The relation $\overset{*}{=}$ is transitive in $\mathcal{F}_\zeta$. If $c \leqslant b$, then the proof is analogical.

Corollary 2.1. The relation $\overset{*}{=}$ is an equivalence relation in any aggregable set.

Theorem 2.2. For any aggregable set F and for any $f,g \in F$, if $f \overset{*}{=} g$, then $f \sim g$.

Proof. The proof is analogical to the proof of Theorem 4 in [2].

Corollary 2.2. For any aggregable set F and for any function $f \in F$, $[f]_{\overset{*}{=}} \subseteq [f]_\sim$.

Theorem 2.3. For any aggregable set F and for any $f \in F$ the equivalence class $[f]_{\overset{*}{=}}$ is a computable set.

Proof. The set $[f]_{\overset{*}{=}}$ is Z-injective, because $[f]_{\overset{*}{=}} \subseteq [f]_\sim$ and $[f]_\sim$ is Z-injective. If $g \in ([f]_{\overset{*}{=}})^*$, then by Theorem 5.3 [1] there exists $h \in [f]_{\overset{*}{=}}$ such that $g \in \{h\}^*$. Hence $g \overset{*}{=} h \overset{*}{=} f$ and $g \overset{*}{=} f$. Since $g \in [f]_{\overset{*}{=}}$, we have $([f]_{\overset{*}{=}})^* \subseteq [f]_{\overset{*}{=}}$.

Since $[f]_{\overset{*}{=}} \subseteq ([f]_{\overset{*}{=}})^*$ (see Corollary 5.1 in [1]), we infer that $([f]^*_{\overset{*}{=}}) = [f]_{\overset{*}{=}}$, whence the set $[f]_{\overset{*}{=}}$ is $*$-closed.

Definition 2.2. A set $F \subseteq \mathcal{F}_\zeta$ is called $*$-connected iff

$$(2.2) \qquad \forall_{f,g \in F} \; (f \overset{*}{=} g).$$

By Corollaries 1.2, 2.2 and Definition 2.2 it is possible to write Theorem 2.3 in the following form.

Theorem 2.4. Any aggregable set and any computable set can be represented as a sum of $*$-connected computable sets.

Theorem 2.5. Let F be an aggregable set, $G \subseteq F$ and $f \in G$. The set G is computable and $*$-connected iff $G^* = G$ and $\{f\}^* \subseteq G \subseteq [f]_{\overset{*}{=}}$.

Proof. The proof is analogical to the proof of Theorem 6 in [2].

Corollary 2.3. For any aggregable set F and for any function $f \in F$ the set $\{f\}^*$ is the minimal set, and the set $[f]_{\overset{*}{=}}$ is the maximal set of all computable and $*$-connected sets G such that $G \subseteq F$ and $f \in G$.

3. Basis of aggregable set

Definition 3.1. A set G is called a basis of an aggregable set F iff $F = \bigcup_{f \in G} [f]_{\overset{*}{=}}$ and

$$(3.1) \qquad \forall_{f,g \in G} \left\{ (f \neq g) \Rightarrow [\sim(f = g)] \right\}.$$

Corollary 3.1. If a set G is a basis of an aggregable set F, then $G \subseteq F$.

Let $F/_{\overset{*}{=}}$ be the quotient space of the equivalence relation $\overset{*}{=}$.

Theorem 3.1. Let F be an aggregable set and $G \subseteq F$. The set G is a basis of F iff it has exactly one element in common with every set $[f]_{\overset{*}{=}} \in F/_{\overset{*}{=}}$.

Proof. The proof is analogical to the proof of Theorem 7 in [2].

Theorem 3.2. If G is a basis of an aggregable set F, then for any set $H \subseteq G$ the set H^* is aggregable and H is a basis of H^*.

Proof. If $H \subseteq G$, then $H^* \subseteq G^*$ and since $G \subseteq F$ and $F^* = F$, we have $G^* \subseteq F$ and $H^* \subseteq F$. By Corollary 1.3 the set H^* is aggregable. In H the condition 3.1 is satisfied. If $g \in H^*$, then by Theorem 5.3 in [1] there exists $f \in H$ such that $g \in \{f\}^*$. If $h \in H^*$ and $h \overset{*}{=} g$, then $h \in \{f\}^*$. If $h \notin \{f\}^*$, then there exists $\varphi \in H$ such that $h \in \{\varphi\}^*$ and $\varphi = f$, and by (3.1), $\sim (\varphi \overset{*}{=} f)$, thus $\sim(h \overset{*}{=} g)$. Thus if $[f]_{\overset{*}{=}} \in H^*/_{\overset{*}{=}}$, then $[f]_{\overset{*}{=}} \subseteq \{f\}^*$. By Theorem 2.5 we have $\{f\}^* \subseteq [f]_{\overset{*}{=}}$, thus $\{f\}^* = [f]_{\overset{*}{=}}$. Hence $H^* = \bigcup_{f \in H} [f]_{\overset{*}{=}}$ by Theorem 5.3 in [1].

Theorem 3.3. If G is a basis of an aggregable set F and $H_1 \subsetneq H_2 \subseteq G$, then $H_1^* \subsetneq H_2^*$.

Proof. If $H_1 \subsetneq H_2$, then $H_1^* \subseteq H_2^*$, thus it is sufficient to demonstrate that if $H_1 \subsetneq H_2 \subseteq G$, then $H_1^* \neq H_2^*$. Really if $H_1 \subsetneq H_2$, then there exists $f \in H_2$ such that $f \notin H_1$. By (3.1) for any function $g \in H_1$, we have $\sim(f \overset{*}{=} g)$, thus $f \notin \{g\}^*$, whence $f \notin H_1^*$. Because $f \in H_2$ and $f \notin H_1^*$, hence $H_1^* \neq H_2^*$.

4. Free basis

Definition 4.1. Bases G_1, G_2 of an aggregable set F are called equivalent (in symbols $G_1 \approx G_2$) if $G_1^* = G_2^*$.

Corollary 4.1. Let G_F be the set of all bases of an aggregable set F. The relation $\approx$ is an equivalence relation in G_F.

Definition 4.2. An equivalence class $[G]_{\approx} \in G_F/_{\approx}$ is called a free basis of F and it is denoted by $\tilde{G}$.

Definition 4.3. Let $\tilde{G}_1, \tilde{G}_2 \in G_F/_{\approx}$. We define a relation $<$ in $G_F/_{\approx}$ as follows

$$(4.1) \qquad (\tilde{G}_1 < \tilde{G}_2) \Longleftrightarrow (\tilde{G}_1^* \subsetneq \tilde{G}_2^*).$$

Theorem 4.1. For any aggregable set F the relation $<$ is irreflexive and transitive in $G_F/_{\approx}$.

Proof. If $\tilde{G}_1, \tilde{G}_2 \in G_F/_{\approx}$ and $\tilde{G}_1 < \tilde{G}_2$, then $G_1^* \subsetneq G_2^*$ and $\sim(G_2^* \subseteq G_1^*)$, hence $\sim(\tilde{G}_2 < \tilde{G}_1)$. If $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3 \in G_F/_{\approx}$, $\tilde{G}_1 < \tilde{G}_2$ and $\tilde{G}_2 < \tilde{G}_3$, then $G_1^* \subsetneq G_2^*$ and $G_2^* \subsetneq G_3^*$, hence $G_1^* \subsetneq G_3^*$. So we have $\tilde{G}_1 < \tilde{G}_3$.

Corollary 4.2. For any aggregable set F the relation $\leq$ defined as follows

$$(4.2) \qquad (G_1 \leq G_2) \Longleftrightarrow (\tilde{G}_1 < \tilde{G}_2) \vee (\tilde{G}_1 = \tilde{G}_2)$$

is a partial order relation in $G_F/_{\approx}$.

Theorem 4.2. A free basis $\tilde{G}_0$ of F is the last element in the partially ordered set $(G_F/_{\approx}, \leq)$ if and only if $G_0^* = F$.

Proof. Let $\tilde{G}_0$ be the last element of $(G_F/_{\approx}, \leq)$ and $G_0^* \neq F$. Since G_0 is a basis of F, we have $G_0^* \subseteq F$ and if $G_0^* \neq F$, then $G_0^* \subsetneq F$. Hence there exists $f \in F$ such that $f \notin G_0^*$. By Definition 1.1 there exists exactly one function $\varphi \in G_0$ such that $\varphi \overset{*}{=} f$. Let H_0 denote the set $(G_0 - \{\varphi\}) \cup \{f\}$. H_0 is clearly a basis of F. We have $G_0 = H \cup \{\varphi\}$ and $H_0 = H \cup \{f\}$, where $H = G_0 - \{\varphi\}$. Whence $G_0^* = H^* \cup \{\varphi\}^*$ and $H_0^* = H^* \cup \{f\}^*$ (see [1]). Because for every $\tilde{G} \in G_F/_{\approx}$ we have $G^* \subseteq G_0^*$, we infer that $H_0^* \subseteq G_0^*$, hence $\{f\}^* \subseteq \{\varphi\}^*$ and $f \in \{\varphi\}^*$. Since $\{\varphi\}^* \subseteq G_0^*$, we have $f \in G_0^*$. Hence if $\tilde{G}_0$ is the last element of $(G_F/_{\approx}, \leq)$, then $G_0^* = F$. If $G_0^* = F$, then for any $\tilde{G} \in G_F/_{\approx}$, we have $G^* \subseteq G_0^*$, because $G^* \subseteq F$.

Definition 4.4. The last element $\tilde{G}_0$ in $(G_F/_{\approx}, \leq)$ is called a proper basis of F.

Corollary 4.3. A free basis $\tilde{G}_0$ is a proper basis of F iff $G_0^* = F$.

Theorem 3.2 implies the following corollary.

Corollary 4.4. If G is a basis of F, then for any set $H \leq G$, H is a proper basis of H^*.

Corollary 4.5. Any aggregable set has at most one proper basis.

5. Free functions

By Definition 3.1 and the condition $\{f\}^* \subseteq [f]_{\stackrel{*}{=}}$ we have the following corollary.

Corollary 5.1. For any function $g \in \{f\}^*$ the set $\{g\}$ is a basis of $\{f\}^*$.

Definition 5.1. Any free basis $\{\tilde{g}\}$ of $\{f\}^*$ is called a free function of $\{f\}^*$.

We denote by $\tilde{g}$ the free function $\{\tilde{g}\}$ of $\{f\}^*$ and by $f/_{\approx}$ we denote the set of all free functions of $\{f\}^*$.

Theorem 5.1. If a function $f \in \mathcal{F}_{\mathcal{C}}$ is Z-injective then any free function $\tilde{g} \in F/_{\approx}$ has exactly one representation g.

Proof. If $\tilde{g} \in f/_{\approx}$, then $g \in \{f\}^*$ and $\{g\}^* \subseteq \{f\}^*$. If $h \in \tilde{g}$, then $\{h\}^* = \{g\}^*$ (see Definitions 5.1 and 4.2). If $\{h\}^* = \{g\}^* \subseteq \{f\}^*$, then $g, h \in \{f\}^*$ and if the function f is Z-injective, then the functions g,h are Z-injective, too (see [1]). If $\{h\}^* = \{g\}^*$, then $h \in \{g\}^*$ and $g \in \{h\}^*$. If $h \in \{g\}^*$, then there exists $a \in T$ such that $h = g_a$, and if $g \in \{h\}^*$, then there exists $b \in T$ such that $g = h_b$. Hence $h = (h_b)_a$. Since the function $h = g_a$ is Z-injective, we obtain $a = b = 0$ and $h = g$.

Theorem 5.2. If a free function $\tilde{g} \in f/_{\approx}$ has more than one representation, then the computable function f is Z-periodic (see [1]).

Proof. By Theorem 5.1 if there exists a free function $\tilde{g} \in f/_{\approx}$ such that $\tilde{g}$ has more than one representation, then f is not Z-injective. Because f is the computable function, thus f is Z-periodic (see Theorem 6.2 in [1]).

Corollary 5.2. If a free function $\tilde{g}$ has more than one representation, then g is Z-periodic.

Theorem 5.3. For any function $f \in \mathcal{F}_\zeta$ the relation $\leqslant$ defined by (4.2) is an ordering relation in $f/_\approx$.

Proof. By Corollary 4.2 the relation $\leqslant$ is a partially ordering relation in $f/_\approx$. It is enough to prove that the relation $\leqslant$ is connective in $f/_\approx$. If $\tilde{g},\tilde{h} \in f/_\approx$ then by Definition 5.1 we have $g,h \in \{f\}^*$ and by Theorem 6.4 from [1], $g \in \{h\}^*$ or $h \in \{g\}^*$. If $g \in \{h\}^*$, then $\{g\}^* \subseteq \{h\}^*$, thus $\tilde{g} \leqslant \tilde{h}$. Analogously, if $h \in \{g\}^*$, then $\tilde{h} \leqslant \tilde{g}$.

Theorem 5.4. For any $f \in \mathcal{F}_\zeta$ the free function $\tilde{f}$ is the last element in $(f/_\approx, \leq)$.

P r o o f. It is enough to demonstrate that for any $\tilde{g} \in f/_\approx$, $\tilde{g} \leq \tilde{f}$. Because $\tilde{g} \in f/_\approx$, thus $g \in \{f\}^*$, hence $\{g\}^* \subseteq \{f\}^*$ and $\tilde{g} \leq \tilde{f}$.

Theorem 5.5. Let F be an aggregable set and $f \in F$. If the set $[f]_{\stackrel{*}{=}} \in F/_{\stackrel{*}{=}}$ has a proper basis $\tilde{G}$, then $\tilde{G}$ is a free function.

Proof. Any basis of $[f]_{\stackrel{*}{=}}$ is a one-element set $\{g\}$. If the set $[f]_{\stackrel{*}{=}}$ has a proper basis, then there exists $g \in [f]_{\stackrel{*}{=}}$ such that $\{g\}^* = [f]_{\stackrel{*}{=}}$ and the free function $\tilde{g}$ is a proper basis of $[f]_{\stackrel{*}{=}}$.

Theorem 5.6. An aggregable set F has a proper basis $\tilde{G}$ iff for every $f \in F$, the set $[f]_{\stackrel{*}{=}}$ has a proper basis $\tilde{g}$.

Proof. If $\tilde{G}$ is a proper basis of F, then for every $f \in F$ there exists exactly one function $g \in G$ such that $g \in [f]_{\stackrel{*}{=}}$. Let $G_1 = G - \{g\}$, thus $G = G_1 \cup \{g\}$ and because $\tilde{G}$ is a proper basis of F, then $G_1^* \cup \{g\}^* = F$. Because $F = \bigcup_{h \in G_1} [h]_{\stackrel{*}{=}} \cup [g]_{\stackrel{*}{=}}$, hence

(5.1) $$G_1^* \cup \{g\}^* = \bigcup_{h \in G_1} [h]_{\stackrel{*}{=}} \cup [g]_{\stackrel{*}{=}} .$$

Because the set $[g]_{\stackrel{*}{=}}$ is separate in G_1^* and $\bigcup_{h \in G_1} [h]_{\stackrel{*}{=}}$, we have

(5.2) $$\{g\}^* \cap [g]_{\stackrel{*}{=}} = [g]_{\stackrel{*}{=}} .$$

By (5.2) we have $[g]_{\stackrel{*}{=}} \subseteq \{g\}$, and because $\{g\}^* \subseteq [g]_{\stackrel{*}{=}}$ (see Theorem 2.5) we obtain $\{g\}^* = [g]_{\stackrel{*}{=}}$. Hence the free function $\tilde{g}$ is a proper basis of $[g]_{\stackrel{*}{=}}$. Since $g \in [f]_{\stackrel{*}{=}}$, we have $[g]_{\stackrel{*}{=}} = [f]_{\stackrel{*}{=}}$, and $\tilde{g}$ is a proper basis of $[f]_{\stackrel{*}{=}}$.

Corollary 5.3. Let F be an aggregable set and G be a basis of F. The set G is a representation of a proper basis $\tilde{G}$ of F if and only if $F = \bigcup_{f \in G} \{f\}^*$ and if any only if $\tilde{G} = \bigcup_{f \in G} \tilde{f}$.

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