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A NUMERICAL SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF TWO NON-LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

In the present paper we give a method of numerical solving of the boundary-value problem for two non-linear second order ordinary differential equations, based on an idea suggested by I.S. Berezin and N.P. Żidkow [1] for one second order non-linear ordinary differential equation. Namely, we approximate differential equations by suitable difference equations and solve the latter system by the method of simple iteratives leading to an open iteration scheme. We prove the existence and uniqueness of the solution of the posed problem, and we show the convergence of the method and estimate the error of the numerical solution.

W. Nikliborc [4] considered the following problem[1)]

$$x'' = f(t,x,y,x',y'), \quad y'' = g(t,x,y,x',y'),$$

$$x(0) = y(0) = 0, \quad [x'(0)]^2 + [y'(0)]^2 = v^2, \; x(\tau) = a, \; y(\tau) = b,$$

where a,b,v,τ are constants, and he showed that under some additional assumption the problem can be solved by the method of succesive approximations.

The problem of the form

$$y_p'' = f_p(x,y_1,y_2,y_1',y_2') \qquad (p=1,2)$$

$$y_p^{(a)} = A_p, \quad y_p^{(b)} = B_p$$

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1) See E. Kamke [3], p. 288.

has been dealt with by G. Scorza-Dragoni [5] who showed that this problem has a solution under the assumption that for $x \in \langle a,b \rangle$ the functions f_p are continuous and bounded.

Now we state the basic problem of this paper. We are given the system

$$(1) \qquad \begin{aligned} x'' &= f(t,x,y,x',y') \\ y'' &= g(t,x,y,x',y') \end{aligned}$$

for $0 \leqslant t \leqslant 1$, with the boundary conditions

$$(2) \qquad \begin{aligned} a_0x(0) + a_1y(0) &= a \\ b_0x'(0) + b_1y'(0) &= b \\ c_0x(1) - c_1y(1) &= c \\ d_0x'(1) - d_1y'(1) &= d. \end{aligned}$$

We represent the problem (1), (2) in the following form

$$(1') \qquad z'' = H(t,z,z')$$

$$(2') \qquad \begin{cases} A\ z(0) + C\ z(1) = \lambda_1 \\ B\ z'(0) + D\ z'(1) = \lambda_2 , \end{cases}$$

where

$$(3) \qquad \begin{cases} z = \begin{bmatrix} x \\ y \end{bmatrix}, \ z' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \ z'' = \begin{bmatrix} x'' \\ y'' \end{bmatrix}, \ H = \begin{bmatrix} f \\ g \end{bmatrix}, \ \lambda_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \ \lambda_2 = \begin{bmatrix} b \\ d \end{bmatrix}, \\ A = \begin{bmatrix} a_0 & a_1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} b_0 & b_1 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ c_0 & -c_1 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 \\ d_0 & -d_1 \end{bmatrix}. \end{cases}$$

We assume that

(a) a_i, b_i, c_i, d_i, $(i=0,1)$, are positive constants

(b) the vector function H is continuous in some convex domain $G \subset R$ with respect to z and z', and has derivatives with respect to t up to the second order continuous and bounded in this domain.

We divide the interval $\langle 0,1\rangle$ into n equal parts with points $0 = t_0 < t_1 < \ldots < t_n = 1$, denoting $t_i - t_{i-1} = \frac{1}{n} = h$ and $z_i = z(t_i)$, $h \in (0, \frac{1}{2})$.

Next we approximate the differential problem (1′), (2′) by the corresponding difference problem

$$(4)\qquad z_{k+1} - 2z_k + z_{k-1} = h^2 H\left(t_k, z_k, \frac{z_{k+1} - z_{k-1}}{2h}\right)(k=1,2,\ldots,n-1)$$

$$(5)\qquad \begin{cases} A\ z_o + C\ z_n = \lambda_1 \\ B(-3z_o+4z_1-z_2) + D(3z_n-4z_{n-1}+z_{n-2}) = 2h\lambda_2 \end{cases}$$

which represents a system of n+1 non-linear algebraic equations with respect to n+1 unknown vector functions z_i. Moreover, the difference problem (4), (5) approximates (1′), (2′) up to h^2.

We shall give an iterative method for solving the system (4), (5). We perform iterations according to the following schema

$$(6)\qquad z_{k+1}^{r+1} - 2z_k^{r+1} + z_{k-1}^{r+1} = h^2 H\left(t_k, z_k^r, \frac{z_{k+1}^r - z_{k-1}^r}{2h}\right) = h^2 H_k^r ,$$

$$(k=1,\ldots,n-1),$$

$$(7)\qquad \begin{cases} A\ z_o^{r+1} + C\ z_n^{r+1} = \lambda_1 \\ B(-3z_o^{r+1}+4z_1^{r+1}-z_2^{r+1}) + D(3z_n^{r+1}-4z_{n-1}^{r+1}+z_{n-2}^{r+1}) = 2h\lambda_2 . \end{cases}$$

The upper indices in (6), (7) denote the number of the consecutive approximation of the problem (1), (2). For a fixed h and the zero approximation $\{z_k^o\}$ the system (6), (7), is a system of linear equations. We assume that for fixed r and

h this system possesses a unique solution which we represent in the form

$$(8) \qquad z_k = u_k + v_k, \quad \text{where} \quad u_k = \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix}, \quad v_k = \begin{bmatrix} v_k^1 \\ v_k^2 \end{bmatrix}.$$

The vector function v_k solves the problem

$$(9) \qquad \begin{cases} v_{k+1}-2v_k+v_{k-1} = [0], \\ A\, v_o + C\, v_n = \lambda_1 \\ B(-3v_o+4v_1-v_2) + D(3v_n-4v_{n-1}+v_{n-2}) = 2h\lambda_2 , \end{cases}$$

and the function u_k solves the problem

$$(10) \qquad \begin{cases} u_{k+1}-2u_k+u_{k-1} = h^2H_k, \\ Au_o + Cu_n = [0], \\ B(-3u_o+4u_1-u_2) + D(3u_n-4u_{n-1}+u_{n-2}) = [0]. \end{cases}$$

T h e o r e m 1. If v_k and u_k are solutions of the problem (9) and (10) respectively, then $z_k = u_k + v_k$, for fixed h and r, solves the problem (6), (7). The proof of this theorem is evident.

The first equation in (9) shows that the second finite difference of the vector function v_k is equal to 0. Hence, we have

$$(11) \qquad v_k = L + F\cdot k \qquad (L \text{ and } F \text{ are column vectors})$$

and v_k $(k=1,2,\dots,n-1)$ satisfies the first equation in (9) for any L and F. We determine L and F from the remaining two equations in (9). After some computations we obtain

$$(A + C)L = \lambda_1 - nCF,$$
$$2(B + D)F = 2h\lambda_2 .$$

By assumption there exist $(A+C)^{-1}$ and $(B+D)^{-1}$. Hence we obtain

$$F = h(B+D)^{-1}\lambda_2,$$

$$L = (A+C)^{-1}\lambda_1 + hn(A+C)^{-1}C(B+D)^{-1}\lambda_2.$$

This yields the following theorem.

T h e o r e m 2. For fixed h and r the system (9) has a unique solution of the form

$$(12)\qquad v_k = (A+C)^{-1}\lambda_1 + h\left[kE - n(A+C)^{-1}C\right](B+D)^{-1}\lambda_2,$$

where E - the 2x2 unit matrix.

Since (9) is a system of linear non-homogeneous equations and has a unique solution, the matrix of this system (conciding with the matrix of (10)) is non-singular. Hence we have the following corollary.

C o r o l l a r y 1. For fixed h and r the system (10), as well as (6), (7), has a unique solution.

For simplicity we write u_k instead of u_k^{r+1}.

We seek the solution of (10) in the form

$$(13)\qquad u_k = h^2\sum_{i=1}^{n-1} g_{ik}H_i, \qquad (k=1,2,\dots,n-1),$$

where g_{ik} is a matrix function which satisfies the system

$$(14)\qquad \begin{cases} g_{ik+1}-2g_{ik}+g_{ik-1} = \begin{cases}[E] & \text{for } i=k\\ [0] & \text{for } i\neq k\end{cases} \\ \qquad\qquad (i=0,\dots,n;\ k=1,\dots,n-1) \\ Ag_{i0} + Cg_{in} = [0] \\ B(-3g_{10}+4g_{i1}-g_{i2}) + D(3g_{in}-4g_{in-1}+g_{in-2}) = [0]. \end{cases}$$

In order that the function (13) satisfy the system (10) it suffices to find a function g_{ik} satisfying (14). Let us consider the equation

(15) $g_{ik+1}-2g_{ik}+g_{ik} = \delta_{ik}E$, ($\delta_{ik}$ is the Kronecker symbol)

equivalent to

(16) $g_{ik+2}-2g_{ik+1}+g_{ik} = \delta_{ik+1}E$, $(i=0,\dots,n;\ k = 0,\dots,n-2)$.

The functions 1, k constitute a fundamental system of the homogeneous equation corresponding to (16). Hence, the general solution of this homogeneous equation has the form

(17) $g_{ik} = \bar{L} + \bar{F}\cdot k$, ($\bar{L}$ and $\bar{F}$ are 2x2 matrices),

We denote the particular solution of (16) by φ_{ik}. It can be represented in the form (see [2] p. 390)

$$\varphi_{ik} = \sum_{q=0}^{k-1} \frac{\begin{vmatrix} 1 & q+1 \\ 1 & k \end{vmatrix}}{\begin{vmatrix} 1 & q+1 \\ 1 & q+2 \end{vmatrix}} \delta_{ik+1}E = \sum_{q=0}^{k-1} \left[k-(q+1)\right] \delta_{ik+1}E .$$

Putting

(18) $$\sigma(i,k) \overset{\text{def}}{=} \sum_{q=1}^{k} \delta_{ik} \qquad \sigma(i,k) = \begin{cases} 1 & \text{for } 1 \leqslant i < k \\ 0 & \text{for } i > k \end{cases}$$

we have

(19) $\varphi_{ik} = \sigma(i,k)(k-1)E$, $(i=0,1,\dots,;\ k=1,2,\dots,n-1)$.

From (18) it follows that

(19′) $$\begin{cases} \sigma(0,k) = 0 \\ \sigma(n,k) = 0 \\ \sigma(i,0) = 0 \end{cases} \qquad (k=1,2,\dots,n-1)$$

By (17) and (19), the general solution of (16), and hence also (15), has the form

(20) $$g_{ik} = \bar{L} + \bar{F}k + \sigma(i,k)(k-i)E .$$

It is easy to verify that the function (20) is the solution of (16). We now determine $\bar{L}$ and $\bar{F}$ in such a way that g_{ik} satisfy the two last equations of (14), namely

(21)
$$A\bar{L} + C\bar{L} + nC\bar{F} + \sigma(i,n)(n-i)C = [0]$$
$$\bar{L} = nM\bar{F} + \sigma(i,n)(n-i)M\,,$$

where $M = -(A+C)^{-1}C$ and

$$2B\bar{F} + B[4\sigma(i,1)(1-i)-\sigma(i,2)(2-i)]+2D\bar{F} +$$
$$+ D[3\sigma(i,n)(n-i)-4\sigma(i,n-1)(n-1-i) + \sigma(i,n-2)(n-2-i)] = [0].$$

Denoting

(22)
$$\begin{cases} g = 4\sigma(i,1)(1-i)-\sigma(i,2)(2-i) \\ f = 3\sigma(i,n)(n-i)-4\sigma(i,n-1)(n-1-i)+\sigma(i,n-2)(n-2-i) \\ l = \sigma(i,n)(n-i) \end{cases}$$

we obtain

(23) $\bar{F} = -\frac{1}{2}(Wg + Nf)$, where $W = -(B+D)^{-1}B$, $N = -(B+D)^{-1}D$.

By (21) - (23) we then have

(24)
$$\bar{L} = \frac{n}{2} M(Wg + Nf) + lM.$$

Hence, the function g_{ik} in (20) has the form

(25)
$$g_{ik} = \begin{cases} \frac{n}{2} M(Wg+Nf)+lM+\frac{1}{2}(Wg+Nf)k + \sigma(i,k)(k-i)E & i \leqslant k, \\ \\ \frac{n}{2} M(Wg+Nf) + lM + \frac{1}{2}(Wg+Nf)k & i \geqslant k. \end{cases}$$

T h e o r e m 3. For a fixed h the function (13) is a solution of system (10).

P r o o f. We take

$$u_{k+1}-2u_k+u_{k-1} = h^2 \sum_{i=1}^{n-1} \left[g_{ik+1}-2g_{ik}+g_{ik-1}\right]H_i .$$

By (25) and (16), the expression in the parantheses under the sign of sum is equal to the unit matrix for $i=k$, and equal to the matrix $[0]$ otherwise, Hence, we have

$$u_{k+1}-2u_k+u_{k-1} = h^2 EH_k = h^2 H_k ,$$

that is the first equation in (10). Further, by (13), (22) and (25) we have

$$Au_o + Cu_n = h^2 \sum_{i=1}^{n-1} \left\{A\left[\frac{n}{2} M(Wg+Nf)+lM\right] + \right.$$

$$\left. + C\left[\frac{n}{2} (Wg+Nf)+lM+\frac{n}{2} (Wg+Nf)+ (i,n)(n-i)E\right]\right\}H_i =$$

$$= h^2 \sum_{i=1}^{n-1} \left\{-\frac{n}{2} C(Wg+Nf)-\sigma(i,n)(n-i)C+\frac{n}{2} C(Wg+Nf)+\sigma(i,n)(n-i)C\right\}H_i=[0].$$

Similarly, we show that the last condition (10) holds. Hence the function (13) is a unique solution of the system (10).

C o r o l l a r y 2. By Corollary 1 and Theorem 1 the unique solution of the system (6), (7) has the form

$$(26) \qquad z_k^{r+1} = (A+C)^{-1}\lambda_1+\left[kE+nM\right](B+D)^{-1}\lambda_2+h^2 \sum_{i=1}^{n-1} g_{ik}H_i^r \qquad (k=0,1,\ldots,n).$$

The iteration formula (26) allows us to compute approximate values of the vector functions $z_k = z(t_k)$ at any point $t_k \in \langle 0,1\rangle$. For a fixed h it suffices to calculate the functions g_{ik} and then to find consecutive approximations of the solution of the system (4), (5). This solution can be also obtained by solving the system (6), (7). The formula (26) allows us to show the convergence of this process and to estimate the error of the numerical solution.

To show convergence we estimate some quantities involved in (25). By the assumptions (a) and (b), we obtain from (25) the following inequalities

$$\left\| h^2 \sum_{i=1}^{n-1} g_{ik} H_i \right\| \leqslant h^2 \sum_{i=1}^{n-1} \left\| g_{ik} \right\| \left\| H_i \right\| \leqslant h^2 \max_i \left\| H_i \right\| \sum_{i=1}^{n-1} \left\| g_{ik} \right\| ,$$

where $\|\cdot\|$ denotes a matrix norm. Since

$$\left.\begin{matrix} g=-1 \\ f=2 \\ l=n-i \end{matrix}\right\} \text{ for } (i=1,2), \quad \left.\begin{matrix} g=0 \\ f=2 \\ l=n-i \end{matrix}\right\} \text{ for } 3\leqslant i\leqslant n-2, \quad \left.\begin{matrix} g=0 \\ f=4 \\ l=n-i \end{matrix}\right\} \text{ for } i=n-1$$

we have

$$\sum_{i=1}^{n-1} \left\| g_{ik} \right\| \leqslant \sum_{i=1}^{2} \left\| \frac{n}{2} M(2N-W)+(n-i)M+\frac{1}{2}(2N-W)k \right\| +$$

$$+\sum_{i=3}^{n-2} \left\| \frac{n}{2} M(2N)+(n-i)M+\frac{1}{2} 2Nk \right\| + \left\| \frac{n}{2} M\cdot 4N+(n-i)M+2Nk \right\| + \frac{k(k-1)}{2} \left\| E \right\| ,$$

$$\text{(27)} \quad h^2 \sum_{i=1}^{n-1} \left\| g_{ik} \right\| \leqslant \left\| MN \right\| + \frac{1}{2} \left\| M \right\| + \frac{3}{2} \left\| W \right\| + \left\| N \right\| + \frac{3}{2} \left\| E \right\| = B_o ,$$

$$\text{(28)} \quad \left\| h^2 \sum_{i=1}^{n-1} g_{ik} H_i \right\| \leqslant B_o \max_i \left\| H_i \right\| ,$$

$$\left\| h^2 \sum_{i=1}^{n-1} \frac{g_{ik+1}-g_{ik-1}}{2h} H_i \right\| \leqslant \frac{h}{2} \max_i \left\| H_i \right\| \sum_{i=1}^{n-1} \left\| g_{ik+1}-g_{ik-1} \right\| ,$$

$$g_{ik+1} = \frac{n}{2} M(Wg+Nf) + lM + \frac{1}{2}(Wg+Nf)(k+1)+\sigma(i,k+1)(k+1-i)E ,$$

$$g_{ik-1} = \frac{n}{2} M(Wg+Nf) + lM + \frac{1}{2}(Wg+Nf)(k-1)+\sigma(i,k-1)(k-1-i)E ,$$

$$g_{ik+1}-g_{ik-1} = Wg+Nf + [\sigma(i,k+1)(k+1-i)-\sigma(i,k-1)(k-1-i)]\,E\ ,$$

$$(29)\qquad \left\| h^2 \sum_{i=1}^{n-1} \frac{g_{ik+1}-g_{ik-1}}{2h} H_i \right\| \leqslant \max_i \|H_i\| \left[2\|N\| + \|E\|\right] = B_1 \max_i \|H_i\|,$$

where B_0, B_1 are constants.

Assume that the vector function $H(t,z,u)$ satisfies the Lipschitz condition with respect to the variables z and u, where $u = \begin{bmatrix} x' \\ y' \end{bmatrix}$ that is, there exist positive constants L_1 and L_2 such that

$$(30)\qquad \|H(t,z,u) - H(t,\bar z,\bar u)\| \leqslant L_1\|z-\bar z\| + L_2\|u-\bar u\| .$$

Let us take the set R^{n+1} consisting of all sequences $\{z_n\} = (z_0,z_1,\dots,z_n)$. For all $\{z_n\}$, $\{\bar z_n\} \in R^{n+1}$ we define a metric by

$$(31)\qquad \varrho(\{z_n\},\{z_n\}) = L_1 \max_k \|z_k-\bar z_k\| + L_2 \max_k \left\| \frac{z_{k+1}-z_{k-1}}{2h} - \frac{z_{k+1}-z_{k-1}}{2h} \right\|.$$

The set R^{n+1} with the so-defined metric is a complete vector space. In the sequel we shall consider only those elements $\{z_k\} \in R^{n+1}$ for which $(z_k,u_k) \in G$, where $u_k = \frac{z_{k+1}-z_{k-1}}{2h}$, $(k=1,\dots,n-1)$. For every element $z_k \in R^{n+1}$ $(k=0,1,\dots,n-1)$ the formula (26) defines a non-linear transformation of the form

$$(32)\qquad \{z_k^{r+1}\} = A(\{z_k^r\})\ ,\quad \text{where } \{z_k^r\} \in R^{n+1},\ \{z_k^{r+1}\} \in R^{n+1} .$$

In the above A denotes the following operation

$$A \overset{\text{def}}{=} (A_0, A_1, \dots, A_n)\ ,$$

$$A_p = S_p + h^2 \sum_{i=1}^{n-1} g_{ip} H_i^r , \quad S_p = (A+C)^{-1}\lambda_1 + h(pE+nM)(B+D)^{-1}\lambda_2.$$

To simplify the notation and to avoid upper indices we define the element $\{\eta_k\}$ as the image of the element $\{z_k\}$. We write this in the form

(33) $$\{\eta_k\} = A(\{z_k\}), \qquad (k=0,1,\dots,n).$$

Having given the elements $\{z_k\} \in R^{n+1}$ we obtain by (26), (28)-(31)

(34) $$\|\eta_k - \bar{\eta}_k\| = \left\| h^2 \sum_{i=1}^{n-1} g_{ik}\left[H(t_i,z_i,u_i) - H(t_i,\bar{z}_i,\bar{u}_i)\right]\right\| \leqslant$$

$$\leqslant h^2 \sum_{i=1}^{n-1} \|g_{ik}\| \left(L_1 \max_k \|z_k - \bar{z}_k\| + L_2 \max_k \|u_k - \bar{u}_k\|\right) \leqslant$$

$$\leqslant B_0 \varrho(\{z_k\}, \{\bar{z}_k\}).$$

Analogously we obtain

(35) $$\left\| \frac{\eta_{k+1}-\eta_{k-1}}{2h} - \frac{\bar{\eta}_{k+1}-\bar{\eta}_{k-1}}{2h} \right\| \leqslant B_1 \varrho(\{z_k\},\{z_k\}).$$

Since B_0 and B_1 are independent of "i" and the inequalities (34) and (35) hold for $k=0,1,\dots,n$, these inequalities hold also for the maximum with respect to k. Multiplying them by L_1 and L_2, respectively, we obtain

$$L_1 \max_k \|\eta_k - \bar{\eta}_k\| \leqslant L_1 B_0 \varrho(\{z_k\},\{\bar{z}_k\}),$$

$$L_2 \max_k \left\| \frac{\eta_{k+1}-\eta_{k-1}}{2h} - \frac{\bar{\eta}_{k+1}-\bar{\eta}_{k-1}}{2h} \right\| \leqslant L_2 B_1 \varrho(\{z_k\},\{\bar{z}_k\})$$

which implies

$$\varrho\left(\{\eta_k\},\ \{\bar{\eta}_k\}\right) \leqslant \gamma\,\varrho\left(\{z_k\},\{\bar{z}_k\}\right), \tag{36}$$

where $\gamma = L_1B_0 + L_2B_1$ independent of h, depends only on a_i, b_i, c_i, d_i $(i=0,1)$. If $\gamma \in (0,1)$, then the map (32) is contractive and the following theorem holds.

T h e o r e m 5. If the zero approximation $\{z_k^0\}$ is such that the points (t_k, z_k^0, u_k^0) belong to G defined in (b), the vector function H satisfies condition (30) and A is a contractive map, then the following inequalities hold

$$\begin{cases} \left\| z_k - z_k^1 \right\| \leqslant \dfrac{B_0}{1-\gamma}\,\varrho\left(\{z_k^1\},\ \{z_k^0\}\right) \\ \left\| u_k - u_k^1 \right\| \leqslant \dfrac{B_1}{1-\gamma}\,\varrho\left(\{z_k^1\},\ \{z_k^0\}\right). \end{cases} \tag{37}$$

P r o o f. These inequalities follow directly from the properties of the map A and from (34) - (36). The set of points (t_k, z_k, u_k) for which (37) holds is a compact set $G_1 \subset G$.

From the principle of contractive maps we obtain the following theorem.

T h e o r e m 6. If in the domain G the function $H(t,z,u)$ is continuous and satisfies condition (30) and A is a contractive map with the constant $\gamma \in (0,1)$, then there exists a unique solution of the system (4), (5) in the form

$$\{z_k\} = \lim_{r\to\infty}\{z_k^r\} \quad \text{for} \quad k=0,1,\ldots,n.$$

This solution can be obtained by the method of successive approximation.

R e m a r k 1. The condition $\gamma \in (0,1)$ is satisfied, when there exist constants $L_1, L_2 \in (0,1)$.

Finally we shall show that the solution of the difference problem (4), (5) is convergent to the solution of the differential problem (1′), (2′) and we shall estimate the error of the numerical solution. To this aim we make the following assumptions:

A. The vector function H satisfies the condition (b) in the domain G.

B. There exists a solution $z = z(t)$ of the problem (1′), (2′) which has bounded derivatives up to the order 4.

C. The difference problem (4), (5) corresponding to the problem (1′), (2′) has a solution with values in the domain G.

Let $\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}$ denote the solution of the problem (1′), (2′) and $\varphi_k = \varphi(t_k)$, $k=0,1,\dots,n$. Besides, let

$$H(t,\varphi(t), \varphi'(t)) = G(t) \quad \text{and} \quad G(t_k) = G_k \tag{38}$$

$$M_j = \max_{t\in\langle 0,1\rangle} \left\| \varphi^{(j)}(t) \right\| , \qquad (j=1,2,3,4). \tag{39}$$

We introduce the column vector

$$\varepsilon_k = \varphi_k - z_k, \qquad (k=0,1,\dots,n),$$

where z_k denotes the approximate solution obtained by the difference method.

Expressing the value $\varphi_{k+1}, \varphi_{k-1}$ by Taylor's formula for the function $\varphi(t)$ in a neighbourhood of the point $t_k \in \langle 0,1 \rangle$ we obtain

$$\varphi_{k+1} = \varphi(t_k+h) = \varphi_k + h\varphi'_k + \frac{h^2}{2!}\varphi''_k + \frac{h^3}{3!}\varphi'''_k + \frac{h^4}{4!}\varphi^{(4)}(t_k+\theta_1 h)$$

$$0 < \theta_1 < 1$$

$$\varphi_{k-1} = \varphi(t_k-h) = \varphi_k - h\varphi_k + \frac{h^2}{2!}\varphi''_k - \frac{h^3}{3!}\varphi'''_k + \frac{h^4}{4!}\varphi^{(4)}(t-\theta_2 h)$$

$$0 < \theta_2 < 1 .$$

This implies

$$(41)\qquad \varphi_{k+1} - 2\varphi_k + \varphi_{k-1} = h^2\varphi''_k + \varrho(k) = h^2G_k + \varrho(k),$$

where

$$(42)\quad \varrho(k) = \frac{h^2}{4!}\left[\varphi^{(4)}(t_k+\Theta_1 h)+\varphi^{(4)}(t_k-\Theta_2 h)\right] \text{ and } \left\|\varrho(k)\right\| \leqslant \frac{1}{12} M_4.$$

In an analogous way we obtain

$$(41')\quad \begin{aligned} &A\varphi_0 + C\varphi_n = \lambda_1 \\ &B(-3\varphi_0+4\varphi_1-\varphi_2)+D(3\varphi_n-4\varphi_{n-1}+\varphi_{n-2}) = 2h\lambda_2+2h^3(B\varrho_0+D\varrho_n), \end{aligned}$$

where

$$(42')\qquad \left\|\varrho_n\right\| = \left\|\varrho_0\right\| \leqslant \frac{1}{3} M_3 .$$

The vector function z_k satisfies the system (4), (5). Hence, by substracting the equations of the system (4), (5) from the corresponding equations of the system (41) i (41') and taking into account (40) we obtain

$$(43)\quad \begin{cases} \varepsilon_{k+1}-2\varepsilon_k+\varepsilon_{k-1} = h^2(G_k-H_k) + h^4\varrho(k), \\ A\varepsilon_0 + C\varepsilon_n = 0, \\ B(-3\varepsilon_0+4\varepsilon_1-\varepsilon_2)+D(3\varepsilon_n-4\varepsilon_{n-1}+\varepsilon_{n-2}) = 2h^3(B\varrho_0+D\varrho_n) = 2h^3 s. \end{cases}$$

The left-hand sides of (41) are analogous to those of (4),(5). By a reasonning analogous to that for the system (4), (5) we obtain the solution

$$(44)\quad \varepsilon_k = h^3(kE+nM)(B+D)^{-1}s+h^2\sum_{i=1}^{n-1} g_{ik}(G_i-H_i)+h^4\sum_{i=1}^{n-1} g_{ik}\varrho(i),$$

$$(k=0,1,\dots,n).$$

(45) $$\|\varepsilon_k\| \leqslant h^3\|kE+nM\|\ \|(B+D)^{-1}\|\cdot\|B\varrho_0+D\varrho_n\| +$$

$$+ h^2\sum_{i=1}^{n-1}\|g_{ik}\|\ \|G_i-H_i\| + h^4\sum_{i=1}^{n-1}\|g_{ik}\|\cdot\|\varrho(i)\|.$$

By assumption, the vector function G_i-H_i satisfies the Lipschitz condition. Further we have

$$h^2\sum_{i=1}^{n-1}\|g_{ik}\|\ \|G_i-H_i\| \leqslant$$

$$\leqslant h^2\sum_{i=1}^{n-1}\|g_{ik}\|\left[L_0\max_i\|\varphi_i-z_i\| + L_1\max_i\left\|\varphi_i' - \frac{z_{i+1}-z_{i-1}}{2h}\right\|\right].$$

Since

$$\left\|\varphi_i - \frac{z_{i+1}-z_{i-1}}{2h}\right\| = \left\|\varphi_i' - \frac{\varphi_{i+1}-\varphi_{i-1}}{2h} + \frac{\varphi_{i+1}-\varphi_{i-1}}{2h} - \frac{z_{i+1}-z_{i-1}}{2h}\right\| =$$

$$= \left\|\varphi_i' - \frac{\varphi_{i+1}-\varphi_{i-1}}{2h} + \frac{\varepsilon_{i+1}-\varepsilon_{i-1}}{2h}\right\| \leqslant h^2\|\mu_1\| + \left\|\frac{\varepsilon_{i+1}-\varepsilon_{i-1}}{2h}\right\|,$$

where

$$\mu_1 = \frac{1}{2\cdot 3!}\left[\varphi'''(\xi_2)+\varphi'''(\xi_1)\right] \quad \text{for } x_{i-1}<\xi_1<x_i,\quad x_i<\xi_2<x_{i+1},$$

and $\|\mu_1\| \leqslant \frac{1}{6}M_3$, we obtain

(46) $$h^2\sum_{i=1}^{n-1}\|g_{ik}\|\ \|G_i-H_i\| \leqslant$$

$$\leqslant h^2\sum_{i=1}^{n-1}\|g_{ik}\|\left(L_0\max_i\|\varepsilon_i\| + L_1\max_i\left\|\frac{\varepsilon_{i+1}-\varepsilon_{i-1}}{2h}\right\| + \frac{1}{6}L_1M_3\right) \leqslant B_0\varrho(\varepsilon,0) +$$

$$+ \frac{h^2}{6}B_0L_1M_3,$$

where $\varrho(\varepsilon,0)$ is the distance between the points $\varepsilon = (\varepsilon_0,\varepsilon_1,\dots,\varepsilon_n)$ and $0 = (\underbrace{0,0,\dots,0}_{n+1})$ of the space R^{n+1}.

From (27) and (42) it follows that

$$h^4 \sum_{i=1}^{n-1} \|g_{ik}\| \, \|\varrho(i)\| \leqslant \frac{h^2}{12} B_0 M_4 . \tag{47}$$

Hence we have

$$\|\varepsilon_k\| \leqslant \frac{h^2}{3} \|E+M\| \, \|(B+D)^{-1}\| \left(\|B\|+\|D\|\right) M_3 + B_0 \varrho(\varepsilon,0) + \frac{h^2}{12} (2L_1 B_0 M_3 + B_0 M_4) . \tag{48}$$

By (44) and (29) we have

$$\varepsilon_{k+1} = h^3 \left[(k+1)E+nM\right](B+D)^{-1} s + h^2 \sum_{i=1}^{n-1} g_{ik}(G_i-H_i) + h^4 \sum_{i=1}^{n-1} g_{ik}\varrho(i),$$

$$\varepsilon_{k-1} = h^3 \left[(k-1)E+nM\right](B+D)^{-1} s + h^2 \sum_{i=1}^{n-1} g_{ik}(G_i-H_i) + h^4 \sum_{i=1}^{n-1} g_{ik}\varrho(i),$$

$$\left\| \frac{\varepsilon_{k+1}-\varepsilon_{k-1}}{2h} \right\| \leqslant \frac{h^2}{3} \|E\| \, \|(B+D)^{-1}\| \left(\|B\|+\|D\|\right) M_3 + B_1 \varrho(\varepsilon,0) + \frac{h^2}{12} (2L_1 B_1 M_3 + B_1 M_4). \tag{49}$$

The inequalities (48), (49) hold for $k=1,2,\dots,n-1$, hence they also hold for the maximum with respect to k. Multiplying (48) and (49) by L_0 and L_1, respectively, and adding side by side we obtain

$$\varrho(\varepsilon,0) \leqslant \frac{h^2}{3(1-\gamma)} \left(L_0 \|E+M\| + L_1 \|E\|\right) \|(B+D)^{-1}\| \left(\|B\|+\|D\|\right) M_3 + \frac{h^2}{12} \gamma (M_4 + 2L_1 M_3). \tag{50}$$

From (50) it follows that $h \to 0$ implies $\varrho(\varepsilon,0) \to 0$, and by (48) we infer that $\|\varepsilon_k\| \to 0$ with $h \to 0$ as fast as $h^2 \to 0$. The same holds for $\left\|\frac{\varepsilon_{k+1}-\varepsilon_{k-1}}{2h}\right\|$ for $k=1,2,\dots,n-1$. Hence we see that for every k we have $y_k \to \varphi_k$ and $\frac{y_{k+1}-y_{k-1}}{2h} \to \varphi_k'$.

Moreover, if in (48) we substitute the right-hand side of (48) in place of $\varrho(\varepsilon,0)$, we obtain an estimation for the error of the numerical solution.

Theorem 7. If the assumptions of Theorem 5 hold and if the vector function H has in G continuous and bounded dervatives up to the order 2, then the error between the numerical and exact solution of the boundary value problem (1), (2) tends to 0 as fast as h^2.

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