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Stefan Węgrzynowski

REPRESENTATION OF GENERALIZED AFFINE SYMMETRIC SPACES BY S-STRUCTURES

Introduction

The classical definition of affine symmetric space is based on the notion of geodesic symmetry. O. Loos [9] gave another definition of affine symmetric space, an axiomatic one, by introducing the structure of differentiable multiplication on manifolds. Modifying the axioms O. Kowalski has generalized these notions to the so-called tangentially regular s-manifolds [6]. Next in [7] he has given a new definition of generalized affine symmetric space.

In this paper we develop some theory which is important for classification of generalized symmetric affine space. Description of a method of the classification and a full list of generalized symmetric affine spaces (of dimension $n \leqslant 4$) will be published in [11], [12].

The present paper was prepared during the author's practice under Professor Oldřich Kowalski at Karol University in Prague in 1975. The author desires to express his gratitude to him for valuable remarks concerning the problem.

I. Differentiable s-manifolds

Following O. Loos, [9], a symmetric space is defined as a manifold M with a differentiable multiplication μ: $M \times M \longrightarrow M$ written as $\mu(x,y) = x \cdot y$ satisfying the following properties:

1° $x \cdot x = x$

2° $x \cdot (x \cdot y) = y$

3^o $x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z)$

4^o every x has a neighbourhood U such that $x\cdot y = y$ implies $y = x$ for all y in U.

For each $x \in M$ the map $s_x : M \longrightarrow M$ given by $s_x(y) = x\cdot y$ is a diffeomorphism, and it is called the symmetry around x. We have $(s_x)^2$ = identity for each x. One of the basic results of [9] is the following.

T h e o r e m A. Each symmetric space $(M, \{s_x\})$ admits a unique linear connection ∇ which is invariant under all symmetries s_x. The connection ∇ is complete and satisfies $T = 0$, $\nabla R = 0$. The affine manifold (M, ∇) is then a usual affine (globally) symmetric space, and the symmetries s_x, $x \in M$, are the usual geodesic symmetries.

In [6], the above theorem was generalized to more general objects, called tangentially regular s-manifolds.

Following O. Kowalski [6] we define a tangentially regular s-structure on a smooth manifold M as a family $\{s_x\}_{x\in M}$ of diffeomorphisms satisfying the following axioms:

(1) $s_x(x) = x$

(2) the tangent map $(s_x)_{*x} : T_x(M) \longrightarrow T_x(M)$ has no fixed vectors except the null vector

(3) $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$,

(4) the map $(x,y) \longrightarrow s_x(y)$ is smooth.

The diffeomomorphisms s_x, $x \in M$, are called symmetries of M. The pair $(M, \{s_x\})$ is called a tangentially regular s-manifold (or shortly, an s-manifold).

It is easy to see that each symmetric space is a tangentially regular s-manifold. (See [6] for details).

An automorphism of $(M, \{s_x\})$ onto itself is a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi \circ s_x = s_{\phi(x)} \circ \phi$ for each $x \in M$. Let us remark that all symmetries s_x of M are automorphisms.

In [6] the following basic theorem was proved:

T h e o r e m B. Let $(M, \{s_x\})$ be a connected s-manifold. Denote by S the tensor field of type (1.1) given by $S_x = (s_x)_{*x}$ for all $x \in M$. Then

1) There is a unique connection $\tilde{\nabla}$ on M (called the canonical connection) such that $\tilde{\nabla}$ is invariant under all s_x and $\tilde{\nabla} S = 0$. $\tilde{\nabla}$ is complete and has parallel curvature and parallel torsion.

2) The group $\mathrm{Aut}(M)$ of all automorphisms of $(M, \{s_x\})$ is a transitive Lie transformation group, which is a closed subgroup of the full affine transformation group $A(M)$ with respect to $\tilde{\nabla}$. The automorphisms of $(M, \{s_x\})$ are exactly those affine transformations which leave the tensor field S invariant.

3) Let G denote the component of unity of $\mathrm{Aut}(M)$, let o be a fixed point of M, and G_o the corresponding isotropy subgroup. Then the homogeneous space G/G_o is reductive in a canonical way and, under the standard identification $G/G_o \cong M$, the connection $\tilde{\nabla}$ coincides with the canonical connection of the second kind of G/G_o.

From [3], Chapter VI, Theorem 7.7 it follows that each s-manifold $(M, \{s_x\})$ admits a subordinated analytic structure for which the tensor field S and the canonical connection $\tilde{\nabla}$ are analytic.

In the subsequent considerations we shall follow, in part, the Riemannian theory which has been developed in [5]. All s-manifolds in question are supposed to be analytic and connected.

D e f i n i t i o n 1. Two s-manifolds $(M, \{s_x\})$, $(M', \{s'_y\})$ are called isomorphic if there is a diffeomorphism $\phi: M \to M'$ (called isomorphism) such that $\phi \circ s_x = s'_{\phi(x)} \circ \phi$ for each $x \in M$. They are called locally isomorphic if, for every two points $p \in M$, $p' \in M'$, there is a diffeomorphism ϕ of a neighbourhood U of p onto a neighbourhood U' of p' (called local isomorphism) with the following property: For each $x \in U$ there is a neighbourhood $V_x \subset U \cap s_x^{-1}(U)$ such that $\phi \circ s_x = s'_{\phi(x)} \circ \phi$ holds on V_x.

L e m m a 1. Let $(M, \{s_x\})$ be an s-manifold, and $\tilde{\nabla}$ its canonical connection. Let ∇ be another connection defined on an open subset $U \subset M$. Suppose that ∇ is locally invariant with respect to $\{s_x\}$ in the following sense: for each $x \in U$, the restriction of s_x to a neighbourhood $V_x \subset U$ is a local affine transformation of the manifold (U, ∇). Then

$$\tilde{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$$

for all vector fields X,Y defined on U, where I is the Kronecker tensor field. Particularly, if $\nabla S_{|U} = 0$, then ∇ coincides with $\tilde{\nabla}$ on U.

The proof of this lemma is the same as that of Proposition 12, [6]. Namely, let $E = \nabla - \tilde{\nabla}$ be the corresponding difference tensor. We write $E_X Y = \nabla_X Y - \tilde{\nabla}_X Y$. Since both $\tilde{\nabla}$ and ∇ are invariant under s_x, $x \in U$, E is invariant with respect to S: $S(E_X Y) = E_{SX} SY$. Now we check easily that

$$(E_{(I-S)^{-1}X} S)(S^{-1}Y) = E_{(I-S)^{-1}X}\left[S(S^{-1}Y)\right] - S(E_{(I-S)^{-1}X} S^{-1}Y) \doteq$$

$$= E_{(I-S)^{-1}X} Y - E_S\left[(I-S)^{-1}X\right] S(S^{-1}Y) = E_{(I-S)^{-1}X - S(I-S)^{-1}X} Y = E_X Y.$$

Since $\tilde{\nabla} S = 0$, we get finally

$$E_X Y = (E_{(I-S)^{-1}X} S)(S^{-1}Y) = (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$$

which was to be proved.

T h e o r e m 1. Let $(M, \{s_x\})$, $(M', \{s'_y\})$ be two s-manifolds with the canonical connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ respectively, and let $U \subset M$, $U' \subset M'$ be open sets. Then a diffeomorphism $\Phi : U \to U'$ is a local isomorphism of $(M, \{s_x\})$ into $(M', \{s'_y\})$ if and only if Φ is a local affine map of $(M, \tilde{\nabla})$ into $(M', \tilde{\nabla}')$ such that $\Phi(S_{|U}) = S'_{|U'}$ [1].

P r o o f. If $\Phi : U \to U'$ is a local isomorphism, we get first $\Phi(S_{|U}) = S'_{|U'}$. Further, let ∇' denote the Φ-image

[1] The image of a tensor field T with respect to a diffeomorphism Φ will be denoted briefly by $\Phi(T)$.

on U' of the connection $\tilde{\nabla}_{|U}$. Because $\tilde{\nabla}_{|U}$ is locally invariant with respect to $\{s_x\}$ then ∇' is locally invariant with respect to $\{s'_y\}$. Moreover, $\nabla'(S'_{|U'}) = 0$ because $\tilde{\nabla}S_{|U} = 0$. According to Lemma 1, ∇' coincides on U' with the canonical connection $\tilde{\nabla}'$, which proves the "only if" part of the theorem.

Let now $\phi : U \to U'$ be a local affine map of $(M, \tilde{\nabla})$ into $(M', \tilde{\nabla}')$, and suppose $\phi(S_{|U}) = S'_{|U'}$. Choose $x \in U$ and a connected neighbourhood V_x of x such that $V_x \subset U \cap s_x^{-1}(U)$. Then the maps $\phi \circ s_x$, $s'_{\phi(x)} \circ \phi$ restricted to V_x are affine diffeomorphisms of V_x onto a connected neighbourhood $V'_{\phi(x)} \subset U'$. Because $\phi(S_{|U}) = S'_{|U'}$, they have the same tangent map $\lambda_x : M_x \to M'_{\phi(x)}$ at x. Consequently, $\phi \circ s_x$ and $s'_{\phi(x)} \circ \phi$ coincide on V_x, q.e.d.

C o r o l l a r y. Two locally isomorphic simply connected s-manifolds are globally isomorphic.

P r o o f. Consider simply connected s-manifolds $(M, \{s_x\})$, $(M', \{s'_y\})$. Let ϕ_U be a local isomorphism of $U \subset M$ onto $U' \subset M'$. Then ϕ_U is a local affine map of $(M, \tilde{\nabla})$ into $(M', \tilde{\nabla}')$ which maps $S_{|U}$ onto $S'_{|U'}$. According to [3, Chapter VI] ϕ_U can be extended to a global affine map $\phi : (M, \tilde{\nabla}) \to (M', \tilde{\nabla}')$. Because S is parallel with respect to $\tilde{\nabla}$ and S' is parallel with respect to $\tilde{\nabla}'$, then $\phi(S) = S'$ on M. Now we can use the second part of the proof of Theorem 1, where we put $V_x = M$ for each x.

T h e o r e m 2. For every s-manifold $(M, \{s_x\})$ there is a simply connected covering s-manifold $(M', \{s'_u\})$ such that the covering map is, in a neighbourhood of each point of M', a local isomorphism.

P r o o f. Denote by $\tilde{\nabla}$ the canonical connection of $(M, \{s_x\})$. Let $(M', \tilde{\nabla}')$ be a simply connected covering manifold of the affine manifold $(M, \tilde{\nabla})$, $\Pi : M' \to M$ being the covering map. $(M', \tilde{\nabla}')$ is complete and analytic because so is $(M, \tilde{\nabla})$. The tensor fields $\tilde{R}$, $\tilde{R}'$ and also $\tilde{T}$, $\tilde{T}'$ are Π-related. Let S' be the lift of S with respect to Π. Then we get $S'(R') = \tilde{R}'$, $S'(\tilde{T}') = T'$, $\tilde{\nabla}'R' = \tilde{\nabla}'\tilde{T}' = \tilde{\nabla}'S' = 0$.

Hence each linear transformation S'_u, $u \in M'$, gives rise to a unique affine diffeomorphism s'_u such that $(s'_{u*})_u = S'_u$ (see [3], Chapter VI). Because S' is parallel with respect to $\tilde{\nabla}'$ and all s_u $(u \in M')$ are affine maps, each tensor field $s'_{u*}(S')$ is also parallel. Now, because S' and $s'_{u*}(S')$ coincide at the point u, they must coincide everywhere. We have obtained $s'_{u*}(S') = S'$ on M' for each $u \in M'$. The last property means that the tangend maps $(s'_u \circ s'_v)_{*v}$ and $(s'_w \circ s'_u)_{*v}$ with $w = s'_u(v)$ coincide for every $u, v \in M'$. Because the maps $s'_u \circ s'_v$ and $s'_w \circ s'_u$ are both affine diffeomorphisms of $(M', \tilde{\nabla}')$, they must coincide, and we get axiom (3) for the family s'_u. Thus $(M', \{s'_u\})$ is an s-manifold. Finally, the tangent maps $(\Pi \circ s'_u)_{*u}$ and $(s_{\Pi(u)} \circ \Pi)_{*u}$ coincide for each $u \in M'$ and hence the affine maps $\Pi \circ s'_u$ and $s_{\Pi(u)} \circ \Pi$ always coincide. The projection Π is locally a diffeomorphism and thus, in the neighbourhood of each point u, a local isomorphism, q.e.d.

II. Infinitesimal s-manifolds

Let $(M, \{s_x\})$ be an s-manifold, $\tilde{\nabla}$ the canonical connection and $\tilde{R}$, $\tilde{T}$ the curvature tensor field and the torsion tensor field of $\tilde{\nabla}$ respectively. Let o be a fixed point of M, and denote by $V = M_o$ the correspoinding tangent space. Denote by S the tensor field of type (1.1) given by $S_x = (s_x)_{*x}$ for all $x \in M$. I is the Kronecker tensor field.

Theorem 3. The tensor fields S, $\tilde{R}$, $\tilde{T}$ satisfy at the initial point o the following algebraic conditions:

(i) Both S_o, $I_o - S_o$ are non-singular linear transformations of V.

(ii) For every $X, Y \in V$ the endomorphism $\tilde{R}_o(X,Y)$ acting are derivation on the tensor algebra $\mathcal{T}(V)$ satisfies $\tilde{R}_o(X,Y)S_o = 0$, $\tilde{R}_o(X,Y)\tilde{R}_o = 0$, $\tilde{R}_o(X,Y)\tilde{T}_o = 0$.

(iii) The tensor $\tilde{R}_o$ and $\tilde{T}_o$ are invariant by S_o.

(iv) $\tilde{R}_o(X,Y) = -\tilde{R}_o(Y,X)$, $\tilde{T}_o(X,Y) = -\tilde{T}_o(Y,X)$.

(v) The first Bianchi identity holds: $\sigma[\tilde{R}_o(X,Y)Z - \tilde{T}_o(T_o(X,Y),Z)] = 0$.

(vi) The second Bianchi identity holds: $\sigma[\tilde{R}_o(T_o(X,Y),Z)] = 0$.

P r o o f. (i) and (iv) are obvious, (ii) and (iii) follows from Part 1) of Theorem B. Finally, (v) and (vi) are the Bianchi identities for the case $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$.

Now we shall prove that the tensor S_o, $\tilde{R}_o$, $\tilde{T}_o$ locally characterize the corresponding s-manifold.

D e f i n i t i o n 2. An infinitesimal s-manifold is a collection $(V,S_o,\tilde{R}_o,\tilde{T}_o)$ where V is a real vector space and $S_o,\tilde{R}_o,\tilde{T}_o$ are tensor of types (1,1), (1,3), (1,2) respectively such that the conditions (i) - (vi) of Theorem 3 are satisfied.

Two infinitesimal s-manifolds $(V_i, S_i, \tilde{R}_i, \tilde{T}_i)$, $i=1,2$, will be called isomorphic if there is a linear isomorphism $f:V_1 \to V_2$ of vector spaces such that $f(S_1) = S_2$, $f(\tilde{R}_1) = \tilde{R}_2$, $f(\tilde{T}_1) = \tilde{T}_2$.

Because the group Aut(M) (see Theorem B) acts transitively on a connected s-manifold $(M, \{s_x\})$ and leaves the tensor fields S, $\tilde{R}$ and $\tilde{T}$ invariant, we see that for every two points $p, q \in M$ the collection $(M_p, S_p, \tilde{R}_p, \tilde{T}_p)$, $(M_q, S_q, \tilde{R}_q, \tilde{T}_q)$ are isomorphic infinitesimal s-manifolds. Therefore, we can introduce the following definition.

D e f i n i t i o n 3. The infinitesimal type of an s-manifold $(M, \{s_x\})$ is the isomorphism class of infinitesimal s-manifolds $(M_p, S_p, \tilde{R}_p, \tilde{T}_p)$, $p \in M$.

T h e o r e m 4. Two s-manifolds $(M, \{s_x\})$, $(M', \{s'_y\})$ are locally isomorphic if and only if they have the same infinitesimal type.

P r o o f. It follows immediately from Theorem 1 that locally isomorphic s-manifolds have the same inifitesimal type. Let now $p \in M$, $p' \in M'$ be two points and assume that there exists an isomorphis f of $(M_p, S_p, \tilde{R}_p, \tilde{T}_p)$ onto $(M'_{p'}, S'_{p'}, \tilde{R}'_{p'}, \tilde{T}'_{p'})$. The affine manifolds $(M, \tilde{\nabla})$, $(M' \tilde{\nabla}')$ have parallel curvature and parallel torsion. According to Theorem 7.4, Chapter VI of [3], there is a affine map

F of a connected neighbourhood U of p onto a neighbourhood U' of p' such that $F(p) = p'$, $F_{*p} = f$. Moreover, $f(S_p) = S'_{p'}$ and $\widetilde{\nabla}S = \widetilde{\nabla}'S' = 0$. Hence $F_*(S_{|U}) = S'_{|U'}$. According to Theorem 1, F is a local isomorphism, q.e.d.

T h e o r e m 5. Each infinitesimal s-manifold $(V, S_o, \widetilde{R}_o, \widetilde{T}_o)$ defines the infinitesimal type of a simply connected s-manifold $(M, \{s_x\})$ which is unique up to an isomorphism.

P r o o f. Let $(V, S_o, \widetilde{R}_o, \widetilde{T}_o)$ be an infinitesimal s-manifold. Let $\underline{h}$ be the Lie algebra of all endomorphism A of V which, as derivations of the tensor algebra $\mathcal{T}(V)$, satisfy $A(S_o) = 0$, $A(\widetilde{R}_o) = 0$, $A(\widetilde{T}_o) = 0$.

Particularly, we have $\widetilde{R}_o(X,Y) \in \underline{h}$ for every $X,Y \in V$ (see axiom (ii)). Following a construction of K. Nomizu, [10], we define a Lie algebra $\underline{g}$ to be the direct sum $V + \underline{h}$ with the multiplication given by

$$(5) \qquad \begin{cases} [X,Y] = (-\widetilde{T}_o(X,Y), -\widetilde{R}_o(X,Y)) \\ [A,X] = AX, \ [X,A] = -AX \\ [A,B] = AB - BA \end{cases}$$

for $X,Y \in V$; $A,B \in \underline{h}$.
One can check easily that the Jacobian identities follow from the conditions (v) and (vi) of Theorem 3.

Let G be the simply connected Lie group with the Lie algebra $\underline{g}$, and H be the connected Lie subgroup corresponding to the Lie algebra $\underline{h} \subset \underline{g}$. H is a closed subgroup of G [10]. Because $[V,\underline{h}] \subset V$ the group G acting on the factor set G/H by left translations is almost effective. G/H is a homogeneous manifold which is simply connected and reductive with respect to the decomposition $\underline{g} = V + \underline{h}$. Denote by $\widetilde{\nabla}$ the canonical connection of the second kind of G/H. Similarly as in the proof of Theorem 8, [5], we identify first $\underline{g} = V + \underline{h}$ with the tangent space G_e and then V with the tangent space $(G/H)_o$ at the origin of G/H via the projection $\Pi: G \longrightarrow G/H$. Starting from $S_o, \widetilde{R}_o, \widetilde{T}_o$ we can construct (in a unique way) tensor fields $S, \widetilde{R}, \widetilde{T}$ on G/H which are G-invariant, and also parallel with respect to $\tilde{\nabla}$. Then there is a family $\{s_x\}$ of affine transformations of $(G/H, \tilde{\nabla})$ uniquely determined by S. Here $(G/H, \{s_x\})$ is an s-manifold for which $\tilde{\nabla}$ is the canonical connection. Also, we can show that $\tilde{R}$ and $\tilde{T}$ are the curvature tensor field and the torsion tensor field of $\tilde{\nabla}$ respectively. Hence we deduce that our s-manifold has the prescribed infinitesimal type. (See [5] for some more details). The uniqueness follows from Corollary of Theorem 1.

T h e o r e m 6. The construction described in Theorem 5 has the same outcome (i.e., it produces the same simply connected s-manifold) if we replace the Lie algebra $\underline{h}$ by its subalgebra $\underline{h}'$ provided that $R_o(X,Y) \in \underline{h}'$ for every $X,Y \in V$.

P r o o f. Let $M = G/H$ be the homogeneous space constructed in Theorem 5. Then G acts almost effectively on M by left translations. Now, $\underline{g}' = V + \underline{h}'$ is a subalgebra of $\underline{g}$; let $G' \subset G$ be the corresponding connected subgroup. Then a standard argument shows that G' acts transitively (and almost effectively) on M. Thus the subgroup $H' \subset H$ corresponding to $\underline{h}'$ is the maximal connected subgroup of G' leaving the origin o fixed. Hence H' is closed in G'.

Now, let $\tilde{G}'$ be a simply connected Lie group with the Lie algebra $\underline{g}'$. Then we can consider $\tilde{G}'$ as the universal covering group of G', and the connected subgroup $\tilde{H}' \subset \tilde{G}'$ corresponding to $\underline{h}'$ covers $\tilde{H}'$. Hence it follows that $\tilde{H}'$ is closed in $\tilde{G}'$. From now on we can proceed as in the second part of the proof of Theorem 5.

R e m a r k. If we take $\underline{h}$ as in the proof of Theorem 5, then the group G is locally isomorphic to the automorphism group of $(M, \{s_x\})$.

If we take the Lie subalgebra $\underline{h}' \subset \underline{h}$ generated by all curvature transformations $\tilde{R}_o(X,Y)$, $X,Y \in V$, then the group G' is locally isomorphic to the transvection group of $(M, \tilde{\nabla})$ (see the next paragraph).

III. Generalized affine symmetric spaces

In this paragraph, we shall present some definitions and results from [7].

Following O. Kowalski, [7], a connected affine manifold $(M, \tilde{\nabla})$ is called a generalized affine symmetric space (shortly: g.a.s. space) if M admits at least one tangentially regular s-structure $\{s_x\}$ such that $\tilde{\nabla}$ is its canonical connection. (An s-structure with this property will be called admissible).

It follows from Theorem B that each g.a.s. space $(M, \tilde{\nabla})$ is a homogeneous and complete affine manifold. Further, from Theorem A we can see that the usual affine symmetric spaces are those g.a.s. spaces which admit a tangentially regular s-structure $\{s_x\}$ with $(s_x)^2$ = identity.

The group of transvections of a generalized affine symmetric space $(M, \tilde{\nabla})$ is the group $Tr(M)$ of all affine transformations φ of $(M, \tilde{\nabla})$ with the following property:

For each $x \in M$ the tangent map $\varphi_{*x} : M_x \to M_{\varphi(x)}$ coincides with the parallel transport along a broken geodesic from x to $\varphi(x)$.

The following theorem shows the connection between the transvection group $Tr(M)$ and the admissible s-structures on $(M, \tilde{\nabla})$.

T h e o r e m C. The transvection group $Tr(M)$ is a connected Lie subgroup of $A(M)$ acting transitively on $(M, \tilde{\nabla})$. For each admissible s-structure $\{s_x\}$ on M, $Tr(M)$ is generated by all transformations of the form $s_x \circ s_y^{-1}$, $x, y \in M$, and it is a normal subgroup of the corresponding automorphism group $Aut(M, \{s_x\})$.

Finally, the Lie algebra $\underline{t}$ of $Tr(M)$ can be obtained in the following way: let V be the tangent space at an arbitrary point $o \in M$, and let $\underline{h}'$ be the Lie subalgebra of $\underline{gl}(V)$ generated by all curvature transformations $\tilde{R}_o(X, Y)$, $X, Y \in V$. Then $\underline{t} = V + \underline{h}'$ with the multiplication given as in Formula (5).

It is easy to see that a g.a.s. space $(M, \widetilde{\nabla})$ satisfies $\widetilde{T} = 0$ if and only if it is locally symmetric. On the other hand, we have the following theorem.

T h e o r e m D. A generalized affine symmetric space $(M, \widetilde{\nabla})$ satisfies $\widetilde{R} = 0$ if and only if $\dim \mathrm{Tr}(M) = \dim M$. Moreover, if $\widetilde{R} = 0$, then the group $\mathrm{Tr}(M)$ is solvable.

IV. On admissible s-structures

M. Berger, [1] has worked out a complete list of local structures of all affine symmetric spaces admitting a transitive semisimple group of automorphisms. He has set aside the spaces of "solvable" and "mixed" type; for such spaces only a topological structural theorem has been proved.

As Theorem D suggests, in the case of generalized affine symmetric spaces the solvable groups play even more important part than in the classical situation. Hence we can guess that the classification of local structures of g.a.s. spaces is a very difficult problem. In this section we shall develop some technical means which can help us to solve the classification problem for the small dimensions, at least. (The method is similar to that used in the classification of generalized symmetric Riemannian spaces of dimension 3,4 and 5, see [5], [8]).

First of all, we shall limit ourselves to the primitive g.a.s. spaces, i.e. those which are not products of g.a.s. spaces. Secondly, we can represent the local structures of g.a.s. spaces by the simply connected g.a.s. spaces (of Theorem 2).

Finally, for our purpose, it is inevitable to represent g.a.s. spaces by certain s-manifolds first (so that we could use the algebraic characterization given in Theorem 5). Yet, it is not necessary to give a list of all s-manifolds of a given dimension to obtain a complete list of g.a.s. spaces of this dimension. Namely, we are going to show that it is quite sufficient to find certain "privileged" s-manifolds.

D e f i n i t i o n 4. An s-manifold $(M, \{s_x\})$ is called the product of s-manifolds $(M_1, \{s_u^1\})$, $(M_2, \{s_v^2\})$ if $M = M_1 \times M_2$, and if for every (u,v), $(p,q) \in M$ we have $s_{(u,v)}(p,q) = (s_u^1(p), s_v^2(q))$.

$(M, \{s_x\})$ is called reducible or irreducible according to as it is a product or not.

P r o p o s i t i o n 1. Let $(M, \{s_x\}) = (M_1, \{s_u^1\}) \times (M_2, \{s_v^2\})$, and denote by $\tilde{\nabla}, \tilde{\nabla}_1, \tilde{\nabla}_2$ the canonical connections on M, M_1, M_2, respectively. Then $(M, \tilde{\nabla}) = (M_1, \tilde{\nabla}_1) \times (M_2, \tilde{\nabla}_2)$.

P r o o f. We can show by means of Theorem B, 1) that product connection $\tilde{\nabla}_1 \times \tilde{\nabla}_2$ on M is the canonical one.

D e f i n i t i o n 5. An infinitesimal s-manifold $(V, S, \tilde{R}, \tilde{T})$ is called the direct sum of infinitesimal s-manifolds $(V_i, S_i, \tilde{R}_i, \tilde{T}_i)$, $i = 1,2$ if $V = V_1 + V_2$ (direct sum) and $S(X) = \sum_{i=1}^{2} S_i(X_i)$, $\tilde{R}(X,Y)Z = \sum_{i=1}^{2} \tilde{R}_i(X_i,Y_i)Z_i$ $T(X,Y) = \sum_{i=1}^{2} \tilde{T}_i(X_i,Y_i)$, where the indices denote the corresponding components of a vector with respect to the decomposition $V = V_1 + V_2$.

An infinitesimal s-manifold $(V,S,\tilde{R},\tilde{T})$ is called reducible or irreducible according to as it is a direct sum or not.

P r o p o s i t i o n 2. A simply connected s-manifold $(M, \{s_x\})$ is reducible if and only if its infinitesimal type is reducible.

The proof follows easily from Proposition 1 and Theorem 5.

D e f i n i t i o n 6. Let $(M, \{s_x\})$ be an s-manifold and $(V,S_o,\tilde{R}_o,\tilde{T}_o)$ its infinitesimal type. The symmetries s_x are called semi-simple if S_o is completely reducible on the complexification V^c of V. The eigenvalues of the s-structure $\{s_x\}$ are defined to be the eigenvalues of S_o.

T h e o r e m 7. Let M be a simply connected manifold. Each admissible s-structure $\{s_x\}$ on the space $(M, \tilde{\nabla})$

can be replaced by an admissible s-structure $\{s'_x\}$ having the same eigenvalues (including the multiplicity) and such that the symmetries s'_x are semi-simple.

P r o o f. Let $\{s_x\}$ be an arbitrary admissible s-structure on $(M, \widetilde{\nabla})$; let $(V, S_o, \widetilde{R}_o, \widetilde{T}_o)$ be the infinitesimal type of $(M, \{s_x\})$. Consider the Lie algebra $\underline{g} = V + \underline{h}$ defined by formula (5). According to the proof of Theorem 5, the affine manifold $(M, \widetilde{\nabla})$ is uniquely determined by $\underline{g}, \underline{h}$ and V. Let us decompose the complexiffication V^c into the eigenspaces corresponding to mutually different eigenvalues of S_o, say $V^c = \sum_{\alpha \in \mathcal{J}} V_{(\alpha)}$. (We always have $\alpha \neq 0,1$). Thus, each $V_{(\alpha)}$ is the subspace of all $Z \in V^c$ such that $(S_o - \alpha I)^k Z = 0$ for some k. Consider the automorphism S_+ of the Lie algebra $\underline{g}$ defined by $S_+ = S_o + id_{\underline{h}}$. The we can write, with respect to S_+, $\underline{g}^c = \sum_{\alpha \in \mathcal{J}} V_{(\alpha)} + V_{(1)}$, where $V_{(1)} = \underline{h}^c$. Now, it is well-known that $[V_{(\alpha)}, V_{(\beta)}] \subset V_{(\alpha \cdot \beta)}$ whenever $\alpha \cdot \beta$ is an eigenvalues of S_+, and $[V_{(\alpha)}, V_{(\beta)}] = 0$ otherwise (Cf. [2], p. 26). Hence we get the following relations:

$$
(6) \quad \begin{cases}
\widetilde{T}_o(V_{(\alpha)}, V_{(\beta)}) = 0 & \text{if } \alpha, \beta \in \mathcal{J},\ \alpha \cdot \beta \notin \mathcal{J} \\
\widetilde{T}_o(V_{(\alpha)}, V_{(\beta)}) \subset V_{(\alpha\beta)} & \text{if } \alpha, \beta,\ \alpha \cdot \beta \in \mathcal{J} \\
\widetilde{R}_o(V_{(\alpha)}, V_{(\beta)}) = 0 & \text{if } \alpha, \beta \in \mathcal{J},\ \alpha \cdot \beta \neq 1 \\
\widetilde{R}_o(V_{(\alpha)}, V_{(\beta)}) \subset \underline{h}^c & \text{if } \alpha, \beta \in \mathcal{J},\ \alpha \cdot \beta = 1 \\
A(V_{(\alpha)}) \subset V_{(\alpha)} & \text{if } A \subset \underline{h}^c.
\end{cases}
$$

Let us define a new linear transformation S'_o on V^c as follows:

$$S_o(Z) = \alpha, Z \quad \text{whenever } Z \in V_{(\alpha)}, \quad \alpha \in \mathcal{J}$$

Because S_o is a real transformation, we have $\bar{Z} \in V_{(\bar{\alpha})}$ for $Z \in V_{(\alpha)}$ and hence S'_o is a real transformation as well.

Further more, S'_o is completely reducible on V^c because all vectors of the eigenspaces $V_{(\alpha)}$, $\alpha \in J$ are eigenvectors of S'_o. Now, from relation (6) we can see that $\widetilde{T}_o$ and $\widetilde{R}_o$ are invariant with respect to S'_o, and $\underline{h} \subset \underline{h}'$, where $\underline{h}'$ is the subalgebra of all endomorphism A of V such that $A(S'_o) = 0$, $A(\widetilde{R}_o) = 0$, $A(\widetilde{T}_o) = 0$. Hence $(V, S'_o, \widetilde{R}_o, \widetilde{T}_o)$ is an infinitesimal s-manifold.

Let $(M', \{s'_x\})$ be the corresponding simply connected s-manifold as in Theorem 5. Here for our construction we have used the Lie algebra $\underline{g}' = V + \underline{h}'$. According to Theorem 6 (where we interchange the meaning of $\underline{h}$ and $\underline{h}'$) we come to the same s-manifold $(M', \{s'_x\})$ using in our construction the Lie algebra $\underline{g} = V + \underline{h}$. On the other hand, we have seen that the Lie algebra $\underline{g} = V + \underline{h}$ determines the affine manifold $(M, \widetilde{\nabla})$. Hence $M' = M$, and the s-structure $\{s'_x\}$ determines the canonical connection $\widetilde{\nabla}$. This completes the proof.

D e f i n i t i o n 7. Let $\mathcal{A}^n$ be the set of all n-tuples (θ_i) of complex numbers such that

a) $\theta_i \neq 0,1$ for $i = 1,\dots,n$

b) there is a permutation ϱ of the indices $1,\dots,n$ such that ϱ^2 = identity and $\theta_{\varrho(i)} = \bar{\theta}_i$ for $i = 1,\dots,n$.

The elements $(\theta_i) \in \mathcal{A}^n$ will be called systems of eigenvalues.

Let us remark that the condition b) is equivalent to the following: if θ is among $\theta_1,\dots,\theta_n$ with the multiplicity m, then so is $\bar{\theta}$ (the reality condition). It is obvious that a family of all eigenvalues of an n-dimensional s-structure belongs to $\mathcal{A}^n$. Such a family will be called a system of eingenvalues of the s-structure. Thus, a system of eigenvalues of an s-structure is uniquely determined up to a permutation.

D e f i n i t i o n 8. The following relations: $\theta_i \cdot \theta_j = \theta_k$, $i \neq j$; $\theta_r \cdot \theta_s = 1$; $\theta_r = \bar{\theta}_s$ satisfied by the numbers $\theta_1,\dots,\theta_n$, $(\theta_1,\dots,\theta_n) \in \mathcal{A}^n$ are called the characteristic relations. We shall denote by $\Sigma(\theta_i)$ the set of all characteristic relations satisfied by an element (θ_i) of $\mathcal{A}^n$.

L e m m a 2. Each relation of the form $\theta_i = \theta_j$, $i \neq j$, is an algebraic consequence of the characteristic relations of $\Sigma(\theta_i)$.

P r o o f. We always have characteristic relations of the form $\theta_{\varrho(i)} = \bar{\theta}_i$, $\theta_{\varrho(j)} = \bar{\theta}_j$ where ϱ is a permutation. Now, $\theta_i = \theta_j$ yields the new characteristic relations $\theta_{\varrho(i)} = \bar{\theta}_j$, $\theta_{\varrho(j)} = \bar{\theta}_i$. Consequently, the equality $\theta_i = \theta_j$ is an algebraic consequence of the characteristic relations $\theta_{\varrho(i)} = \bar{\theta}_i$, $\theta_{\varrho(j)} = \bar{\theta}_j$, $\theta_{\varrho(i)} = \bar{\theta}_j$, $\theta_{\varrho(j)} = \bar{\theta}_i$.

D e f i n i t i o n 9. A system of eigenvalues $(\theta_i) \in \mathcal{A}^n$ is called reducible if there is an index $k < n$ and a permutation σ of $1,\dots,n$ such that $\Sigma(\theta_1,\dots,\theta_n) = \Sigma(\theta_{\sigma(1)},\dots,\theta_{\sigma(k)}) \cup \Sigma(\theta_{\sigma(k+1)},\dots,\theta_{\sigma(n)})$ (Here $(\theta_{\sigma(1)},\dots,\theta_{\sigma(k)})$ and $(\theta_{\sigma(k+1)},\dots,\theta_{\sigma(n)})$ are considered as elements of $\mathcal{A}^k$ and $\mathcal{A}^{n-k}$ respectively).

T h e o r e m 8. If a simply connected s-manifold $(M, \{s_x\})$ possesses a reducible system of eigenvalues (θ_i) then it is reducible.

P r o o f. We can choose our system of eigenvalues in such a way that

$$\Sigma(\theta_1,\dots,\theta_n) = \Sigma(\theta_1,\dots,\theta_k) \cup \Sigma(\theta_{(k+1)},\dots,\theta_n).$$

Consider now a permutation ϱ such that ϱ^2 = identity, and $\theta_{\varrho(i)} = \bar{\theta}_i$ for $i = 1,\dots,n$. Then ϱ decomposes into $\varrho_1 : \{1,\dots,k\} \to \{1,\dots,k\}$ and $\varrho_2 : \{k+1,\dots,n\} \to \{k+1,\dots,n\}$.

Let $(V,S,\tilde{R},\tilde{T})$ be the infinitesimal type of $(M, \{s_x\})$ and let $\{U_1,\dots,U_n\}$ be a basic of V^c such that, for each $i = 1,\dots,n$, $U_i \in V_{\theta_i}$ (= the full eigenspace of θ_i). Moreover, we can suppose that $U_{\varrho(i)} = \bar{U}_i$ for $i = 1,\dots,n$. Then the subspace $V_1^c \subset V^c$ generated by $U_1,\dots,U_k$ and the subspace V_2^c generated by $U_{k+1},\dots,U_n$ are complexifications of certain subspaces V_1 and V_2 of V, respectively. In fact, we have $(\overline{V}_1^c) = V_1^c$, $(\overline{V}_2^c) = V_2^c$ in V^c. According to Lemma 2, we can see that each eigenspace $V_{\theta_i} \subset V^c$ is contained either in V_1^c or in V_2^c. Hence we get a decomposition $V = V_1 + V_2$, where V_1, V_2 are invariant subspaces with respect to S.

From Formula (6) it follows that $\tilde{T}(V_i,V_i) \subset V_i$, $\tilde{R}(V_i,V_i)V_j \subset V_j$ for $i,j=1,2$; and $\tilde{T}(V_1,V_2) = 0$, $\tilde{R}(V_1,V_2)=0$. Thus the infinitesimal s-manifold $(V, S, \tilde{R}, \tilde{T})$ is reducible and consequently, the simply connected s-manifold $(M, \{s_x\})$ is also reducible.

Now we shall define two relations on $\mathcal{A}^n$.

D e f i n i t i o n 10. We write $(\theta_i) \dashv (\theta_i')$ if and only if $\Sigma(\theta_i) \subseteq \Sigma(\theta_i')$ after a possible re-numeration of the numbers θ_i'. Further, denote $(\theta_i) \sim (\theta_i')$ if and only if $(\theta_i) \dashv (\theta_i')$ and $(\theta_i') \dashv (\theta_i)$.

T h e o r e m 9. Let M be a simply connected n-dimensional manifold. If the space $(M, \tilde{\nabla})$ admits an s-structure $\{s_x\}$ with a system of eigenvalues (θ_i), and if $(\theta_i') \vdash (\theta_i)$ in $\mathcal{A}^n$, then $(M, \tilde{\nabla})$ admits an s-structure $\{s_x'\}$ with the system of eigenvalues (θ_i).

P r o o f. Let $(V,S,\tilde{R},\tilde{T})$ denote the infinitesimal type of $(M, \{s_x\})$. We can re-numerate the numbers (θ_i') in such a way that $\Sigma(\theta_i) \subset \Sigma(\theta_i')$. We can also suppose that S is completely reducible (Theorem 7). Let ϱ be a permutation such that $\varrho^2 = \mathrm{id}$ and $\theta_{\varrho(i)} = \overline{\theta}_i$ for $i = 1,\dots,n$. Then V^c possesses a basis $U_1,\dots,U_n$ of eigenvectors corresponding to $\theta_1,\dots,\theta_n$, respectively, and such that $U_{\varrho(i)} = \overline{U}_i$, $i = 1,\dots,n$.

Let us define a transformation S_o' of V^c by $S_o'U_i = \theta_i' U_i$ for $i = 1,\dots,n$. First of all, the relations $\theta_{\varrho(i)} = \overline{\theta}_i$ are characteristic relations and hence $\theta'_{\varrho(i)} = \overline{\theta}_i'$ for each i.

Then $S_o'\overline{U}_i = S_o'U_{\varrho(i)} = \theta'_{\varrho(i)}U_{\varrho(i)} = \theta'_iU_{\varrho(i)} = \overline{\theta'_i}\overline{U}_i = \overline{\theta_iU_i} = \overline{S_o'U_i}$. i.e, S_o' is a real transformation. Further, let $\tilde{T}_o(U_i,U_j) = \sum T_{ij}^kU_k$. Because $S_o(\tilde{T}_o) = \tilde{T}_o$, we can have $T_{ij}^k \neq 0$ only if $\theta_i\cdot\theta_j = \theta_k$, or else, only if $\theta'_i\cdot\theta'_j = \theta'_k$. Hence

$$S_o'(\tilde{T}_o(U_i,U_j)) = S_o'\Big(\sum_k T_{ij}^kU_k\Big) = \sum_k T_{ij}^k(S_o'U_k) =$$

$$= \sum T_{ij}^k\theta'_kU_k = \sum T_{ij}^k\theta'_i\theta'_jU_k = \theta'_i\theta'_j\sum T_{ij}^kU_k =$$

$$= \theta'_i\theta'_j\tilde{T}_o(U_i,U_j) = \tilde{T}_o(\theta'_iU_i,\ \theta'_jU_j) = \tilde{T}_o(S_o'U_i,S_o'U_j),$$

i.e $S_o'(\tilde{T}_o) = \tilde{T}_o$.

Now, let A be an endomorphism such that $A(S_o) = 0$. Then A leaves invariant the eigenspaces of S_o corresponding to the mutually different eigenvalues among $\theta_1,\dots,\theta_n$. According to Lemma 2 if θ_i has the multiplicity m, then θ'_i has the multiplicity at least m. Hence each eigenspace of S_o' is a direct sum of certain eigenspaces of S_o. Consequently, A leaves invariant the eigenspaces of S_o' and thus $A(S_o') = 0$.

Particularly, $\tilde{R}_o(X,Y)S_o = 0$ implies $\tilde{R}_o(X,Y)S_o' = 0$. The relations $S_o(\tilde{R}_o) = \tilde{R}_o$ and $\tilde{R}_o(X,Y)S_o = 0$ yield $\tilde{R}_o(S_oX,S_oY) = \tilde{R}_o(X,Y)$. Now, $\tilde{R}_o(U_i,U_j) \neq 0$ only if $\theta_i\cdot\theta_j = 1$, or else, only if $\theta'_i\cdot\theta'_j = 1$. Hence $\tilde{R}_o(S_o'U_i,\ S_o'U_j) = \tilde{R}_o(\theta'_iU_i,\ \theta'_jU_j) = \theta'_i\cdot\theta'_j\tilde{R}_o(U_i,U_j) = \tilde{R}_o(U_iU_j)$, i.e. $\tilde{R}_o(S_o'X,\ S_o'Y) = \tilde{R}_o(X,Y)$, and thus $S_o'(\tilde{R}_o) = \tilde{R}_o$. We conclude that $(V,S_o',\tilde{R}_o,\tilde{T}_o)$ is an infinitesimal s-manifold. For the Lie algebras $\underline{h}$ and $\underline{h}'$ we obtain the inclusion $\underline{h}\subset\underline{h}'$.

Quite similarly as in the proof of Theorem 7 we can show that the simply connected s-manifold $(M',\{s_x'\})$ coressponding to $(V, S_0', \tilde{R}_0, \tilde{T}_0)$ yields the affine manifold $(M, \tilde{\nabla})$. Thus $\{s_x'\}$ is an admissible s-structure on $(M, \tilde{\nabla})$ with the system of eigenvalues (θ_i').

Definition 11. A system of eigenvalues $(\theta_i) \in \mathcal{A}^n$ is called maximal, if for any $(\theta_i') \in \mathcal{A}^n$ the relation $(\theta_i') \leftarrow (\theta_i)$ implies $(\theta_i') \sim (\theta_i)$.

From Theorem 8 we obtain the following corollary.

Corollary. A simply connected generalized affine symmetric space $(M, \tilde{\nabla})$ always admits an s-structure $\{s_x\}$ with a maximal system of eigenvalues.

If $(\theta_i) \in \mathcal{A}^n$ is an "admissible" system of eigenvalues for $(M, \tilde{\nabla})$, then all system $(\theta_i') \in \mathcal{A}^n$ such that $(\theta_i') \sim (\theta_i)$ are admissible for $(M, \tilde{\nabla})$.

Definition 12. An s-structure $\{s_x\}$ on M is called of order k ($k > 1$ being an integer) if $(s_x)^k = id$ for all $x \in M$, and if k is the least integer with this property. If such an integer k does not exist, then $\{s_x\}$ is said to be of infinite order.

Obviously, if (θ_i) is a system of eigenvalues of $\{s_x\}$, then the relation $(s_x)^k = id$, $x \in M$, is equivalent to the relation $(\theta_i)^k = 1$, $i = 1, \ldots, n$.

Definition 13. A generalized affine symmetric space $(M, \tilde{\nabla})$ is called of order k if it admits an s-structure of order k, and it does not admit any s-structure of order $l < k$. $(M, \tilde{\nabla})$ is called of infinite order if it admits only s-structures of infinite order.

Theorem 10. Let $(M, \tilde{\nabla})$ be a generalized affine symmetric space which is simply connected and of finite order. Then $(M, \tilde{\nabla})$ admits a structure $\{s_x\}$ of finite order and with a maximal system of eigenvalues.

To prove this theorem, we shall need some lemmas boforehand.

Lemma 3. Each relation of the form $|\theta_j| = 1$ satisfied by a system of eigenvalues $(\theta_i) \in \mathcal{A}^n$ is an algebraic consequence of the characteristic relations.

P r o o f. Let ϱ be a permutation of $\{1,\dots,n\}$ such that $\theta_{\varrho(i)} = \bar{\theta}_i$ for $i = 1,\dots,n$. Then $|\theta_j| = 1$ implies $\theta_{\varrho(j)}\cdot\theta_j = 1$. Conversely, the characteristic relations $\theta_{\varrho(j)} = \bar{\theta}_j$, $\theta_{\varrho(j)}\cdot\theta_j = 1$ imply $|\theta_j| = 1$.

L e m m a 4. Let (V, g) be a vector space with a positive scalar product, and let us denote by $O((V,g)$ the group of all orthogonal transformations of (V,g). For $S \in O(V,g)$, let $Cl(S)$ denote the least closed subgroup of $O(V,g)$ generated by S. Then for each $S \in O(V,g)$ without non-zero fixed vector there is a periodic transformations $S^* \in Cl(S)$ without non-zero fixed vectors.

Proof is the same as that of Lemma 3, [5].

P r o o f o f T h e o r e m 10. Let $\{s_x\}$ be an admissible s-structure of finite order on $(M, \tilde{\nabla})$. In case that the eigenvalues θ_i of $\{s_x\}$ do not form a maximal system let us consider a maximal system of eigenvalues $(\theta_i') \succ (\theta_i)$. Let $\{s_x'\}$ be a corresponding s-structure on $(M, \tilde{\nabla})$. We can suppose that $\{s_x'\}$ is of infinite order (otherwise the theorem obviously holds).

Since $\{s_x\}$ is of finite order, the system (θ_i) is also of finite order, i.e., $(\theta_i)^k = 1$ for $i = 1,\dots,n$. Particularly, we have $|\theta_i| = 1$ for $i = 1,\dots,n$, and according to Lemma 3 we obtain $|\theta_i'| = 1$ for $i = 1,\dots,n$. Let $(V,S',\tilde{R},\tilde{T})$ be the infinitesimal type of $(M, \{s_x'\})$. Then the least closed subgroup $Cl(S')$ of $GL(V)$ generated by S' is compact, and hence it admits an invariant scalar product g. Obviously, $Cl(S') \subset O(V,g)$. According to Lemma 4, $Cl(S')$ contains a periodic transformation S^* without non-zero fixed vectors. Thus, the corresponding system of eigenvalues (θ_i^*) belongs to $\mathcal{A}^n$ and it is of finite order. It suffices to prove that $(\theta_i^*) \succ (\theta_i')$.

Obviously, in the natural topology of $O(V,g)$, we can write $S^* = \lim_{n\to\infty} (S'^{n_j})$, where $\{n_j\}$ is a subsequence of the sequence 1,2,... Here the sequence $\{n_j\}$ can be selected in such a way that all S'^{n_j} are transformations without non--zero fixed vectors. Consequently, $(\Theta_i'^{n_j}) \in \mathcal{A}^n$ for each n_j. It is obvious that $(\Theta_i'^{n_j}) \prec (\Theta_i')$ for all n_j and hence $(\Theta_i^*) = \lim_{n_j \to \infty} (\Theta_i'^{n_j}) \prec (\Theta_i')$. This completes the proof.

Resuming the results of Theorem 9 and 10, we obtain

T h e o r e m 11. In each dimension n, the set $\mathcal{A}^n$ possesses a finite subset $\mathcal{E}^n$ of s_n elements (s_n = the number of equivalence classes of maximal elements of $\mathcal{A}^n$) with the following property:

Each simply connected generalized affine symmetric space $(M, \tilde{\nabla})$ of dimension n admits an s-structure $\{s_x\}$ with a system of eigenvalues $(\Theta_i) \in \mathcal{E}^n$. Particularly, of $(M, \tilde{\nabla})$ is of finite order, then $\{s_x\}$ can be chosen of finite order.

Applying results of the present paper we shall in [11], [12] the classification of generalized affine symmetric spaces of dimension $n \leqslant 4$. There will be exactly 3 families of g.a.s. spaces of dimension 3 and 15 families of g.a.s. spaces of dimension 4. All of them being simply connected, irreducible and not locally symmetric. We shall show, that g.a.s. spaces of dimension 2 are usual affine symmetric spaces.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, SZCZECIN

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