DEMONSTRATIO MATHEMATICA
Vol. IX No 4 1976

Suhrit K. Dey

NUMERICAL SOLUTION OF A SYSTEM OF NONLINEAR EQUATIONS BY PERTURBATIONS

1. Introduction

The numerical scheme which is often used to solve systems of nonlinear equations is Newton's method. It has a quadratic rate of convergence. But in order that it may converge, initial estimates of the solution are required to be close to the solution. Furthermore, at each iteration, a Jacobian whose value must be other than zero has to be evaluated. These are indeed severe drawbacks of this method especially when a large system of nonlinear equations is to be solved. On the other hand, simpler schemes of iterations are the functional iterations, namely, Jacobi iterations and Gauss-Seidel iterations. These could be easily applied to large systems, but their rates of convergence are linear. In this paper, attempts are made to develop a simple functional iterative scheme having a quadratic rate of convergence with respect to other functional iterations.

The method has been derived by perturbing nonlinear Jacobi iterations. Its effectiveness is established through several applications. Some of them are discussed in section 6. It has been proved mathematically and verified computationally that the perturbation parameters control the mode of convergence of iterations. Also, because of these perturbations, since the iterates undergo displacements, it is expected that in some cases these perturbed iterations may converge to the solution whereas nonlinear Jacobi iterations will fail. This has been verified through examples in section 6.

Various perturbation methods to solve systems of equations can be found in [1] - [4]. Hence the basic concept in this work is not new. However, since the algorithm of this method is much simpler than the previous methods, when applied to a large system of nonlinear equations, it should take less computer time and computer memory storage.

2. Formulation of the algorithm

The problem before us is to solve a system of nonlinear equations

$$(1) \qquad f_i(x_1,x_2,\ldots,x_n) = 0, \qquad i = 1,2,\ldots,n,$$

in a domain D which is a subspace of R^n, where R^n is the n-dimensional real space.

Let x be a vector in D, given by $x = (x_1,x_2,\ldots,x_n)^T$, where the superscript T denotes the transpose of a matrix. Let $F : D \subset R^n \longrightarrow R^n$, then (1) may be expressed as

$$(2) \qquad F(x) = 0.$$

It is assumed that there exists a solution $x = x^* \in D$ of (2), where $x^* = (x_1^*,x_2^*,\ldots,x_n^*)^T$. Thus $F(x^*) = 0$. Let us express the system (1) as

$$(3) \qquad x_i = G_i(x_1,x_2,\ldots,x_n), \qquad i = 1,2,\ldots,n).$$

These equations may be expressed as

$$(4) \qquad x = G(x),$$

where $G : F \subset R^n \longrightarrow R^n$. Since $x = x^*$ is a solution of (4), thus x^* is a fixed point of the operator G.

Let $x^{(0)} \in D$, be an initial estimate of x^*. The nonlinear Jacobi iteration at some k-th iteration may be expressed as

$$(5) \qquad x_i^{(k)} = G_i\left(x_1^{(k-1)},x_2^{(k-1)},\ldots,x_n^{(k-1)}\right), \qquad i = 1,2,\ldots,n,$$

where $x_i^{(k)}$ is the value of x_i at the k-th iteration. We will now introduce a perturbed iterative scheme as follows

$$x_i^{(k)} = w_i^{(k)} + G_i\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_n^{(k-1)}\right), \tag{6}$$

where $W_i^{(k)}$ are the elements of the perturbation vector $W^{(k)} = \left(W_1^{(k)}, W_2^{(k)}, \ldots, W_n^{(k)}\right)^T \in R^n$; and will be computed in terms of quantities obtained at the (k-1)-th iteration.

Let us assume that, for all k, $x^{(k)} \in D$ and $g^{(k)} \in D$, where

$$g^{(k)} = \left(G_1^{(k)}, G_2^{(k)}, \ldots, G_n^{(k)}\right) \text{ with } G_i^{(k)} = G_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right).$$

To compute $W_i^{(k)}$ we will assume that these quantities are small such that terms containing their squares may be neglected. Also the functionals G_i must be such that their first and second derivatives at any k iteration must satisfy the following conditions

$$\left|\frac{\partial G_i}{\partial x_i}\right|^{(k)} \neq 1 \quad \text{and bounded,} \tag{7a}$$

$$\left|\frac{\partial^2 G_1}{\partial x_i^2}\right|^{(k)} \quad \text{bounded} \tag{7b}$$

for $i = 1,2,\ldots,n$, and $k = 1,2,\ldots$ If the iterative scheme (6) converges to the solution x^* after k-1 iterations, $x_i^{(k)} = x_i^{(k-1)} = x_i^*$ for $i = 1,2,\ldots,n$.
Thus

$$x_i^{(k)} = G_i\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_{i-1}^{(k-1)}, x_i^{(k)}, x_{i+1}^{(k-1)}, \ldots, x_n^{(k-1)}\right)$$

or,

$$W_i^{(k)} + G_i^{(k-1)} =$$

$$= G_i\left(x_1^{(k-1)}, x_2^{(k-1)}, \ldots, x_{i-1}^{(k-1)}, W_i^{(k-1)} + G_i^{(k-1)}, x_{i+1}^{(k-1)}, \ldots, x_n^{(k-1)}\right).$$

Expanding the right-hand side of this equation by Taylor's series, and truncating after the second term by virtue of the assumptions imposed upon $W_i^{(k)}$ and (7b) on G_i, we have

$$(8) \qquad W_i^{(k)} = \frac{G_i\left(x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}, G_i^{(k-1)}, x_{i+1}^{(k-1)}, \ldots, x_n^{(k-1)}\right) - G_i^{(k-1)}}{1 - \left(\frac{\partial G_i}{\partial x_i}\right)_{x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}, G_i^{(k-1)}, x_{i+1}^{(k-1)}, \ldots, x_n^{(k-1)}}}.$$

The equations (6) and (8) form the required algorithm which we will apply to solve systems of nonlinear equations numerically.

3. Convergence theorems

We will now analyze the convergence properties of the algorithm. In the vector form the equation (6) may now be expressed as

$$(9) \qquad x^{(k)} = W^{(k)} + G\left(x^{(k-1)}\right).$$

(cf. [4] section 13.1.4), may now be developed as follows Let $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$.

T h e o r e m 1: Let $G : D \subset R^n \to R^n$ be such that,

$$(10) \qquad |G(Y) - G(z)| \leqslant A|y-z|$$

$\forall\, y, z \in D$, where A is an isotone matrix $(n \times n)$ and its spectral radius $\varrho(A) < 1$. Furthermore, if

$$(11) \qquad \lim_{k\to\infty} |W^{(k)}| = 0$$

the iterative scheme (9) will converge to the solution x^* and x^* is the unique solution in D.

P r o o f: $|x^{(k)} - x^*| \leqslant |W^{(k)}| + |G(x^{(k-1)}), G(x^*)| \leqslant$

$\leqslant |W^{(k)}| + A|x^{(k-1)} - x^*| \leqslant \ldots \leqslant \sum_{j=1}^{k} A^{k-j}|W^{(j)}| + A^k|x^{(0)} - x^*|.$

Since $\varrho(A) < 1$, we can choose a norm such that $\|A\| < 1$. Let $e^{(k)} = |x^{(k)} - x^*|$ and $v^{(j)} = |W^{(j)}|$. From (11) for some $j \geqslant k_0$, $\|v^{(j)}\| < \varepsilon$. Thus

$$\|e^{(k)}\| \leqslant \|A^k e^{(o)}\| + \sum_{j=0}^{k} \|A^{k-j} v^{(j)}\| + \sum_{j=k_0+1}^{k} \|A^{k-j} v^{(j)}\| \leqslant$$

$$\leqslant \|A\|^k \|e^{(o)}\| + \|A\|^{k-k_o} \|v^{(k_o)}\| + \varepsilon \frac{\|A\|^{k_o+1}}{1-\|A\|}.$$

Since $\|A\| < 1$, as $k \to \infty$, $\|e^{(k)}\| \to 0$. This establishes convergence of iterations to the solution.

To prove uniqueness, let us assume that y^* is another fixed point of G in D. Then,

$$|x^* - y^*| = |G(x^*) - G(y^*)| \leqslant A|x^* - y^*|.$$

Thus, $(I - A)|x^* - y^*| \leqslant 0$. Since $\varrho(A) < 1$, then

$$(I - A)^{-1} = \sum_{p=0}^{\infty} A^p > 0.$$

Hence, $|x^* - y^*| \leqslant 0$. Therefore, $x^* = y^*$.

It is rather easy to establish that the condition (11) is a necessary condition for convergence of the iterative scheme (9).

T h e o r e m 2. If the mapping G satisfies (10), the condition (11) is a necessary condition for convergence of the iterative scheme (9) to the solution x^* in D.

P r o o f: From (9)

$$|W^{(k)}| = |x^{(k)} - G(x^{(k-1)})| \leqslant |x^{(k)} - x^*| + A|x^{(k-1)} - x^*|.$$

If $y, z \in R$ are two vectors, $|y| \leqslant |z|$ implies that $\|y\| \leqslant \|z\|$. Thus we get,

$$\|W^{(k)}\| \leqslant \|x^{(k)} - x^*\| + \|A\| \|x^{(k-1)} - x^*\| .$$

If the iterations converge to x^*, for some $k \geqslant k_0 + 1$,

$$\|x^{(k)} - x^*\| \leqslant \varepsilon/2 \text{ and } \|x^{(k-1)} - x^*\| \leqslant \varepsilon/(2\|A\|).$$

Thus, $\|W^{(k)}\| \leqslant \varepsilon$. This proves the theorem.

4. Criterion for convergence of iterations

In most iterative schemes, convergence of iterations is accepted if after some k iterations,

$$|x_i^{(k)} - x_i^{(k-1)}| < \varepsilon \tag{12}$$

for $i = 1,2,\ldots,n$, where ε is a small positive quantity chosen arbitrarily. Although mathematically the validity of this condition is understandable, in a computer program it is merely a necessary condition for convergence. Thus it is conceivable that a slowly converging iterative scheme may satisfy (12) and converge to a wrong solution even when $\varepsilon = 0.0001$.

In the present scheme, the mode of convergence is asymptotic and directly depends on the values of the perturbation vector $W^{(k)}$. This consistent with (11) the convergence criterion is set as

$$\max_i |W_i^{(k)}| < \varepsilon, \tag{13}$$

where (in a computer program) we generally chose $\varepsilon \leqslant 10^{-4}$. If the assumptions imposed upon the operator G are satisfied, then according to the theorems 1 and 2, the condition (13) is both necessary and sufficient for convergence. This was verified by computer experimentations.

5. Computational procedure

Let us consider an illustration showing thereby the computational procedure of the algorithm.

E x a m p l e. Solve: $x_1 - \cos(x_1,x_2) + 1 = 0$, $x_2 - \sin(x_1,x_2) = 0$.

This system has a unique solution: $x_1 = 0$, $x_2 = 0$. Let $G_1(x_1,x_2) = \cos(x_1,x_2) - 1$ and $G_2(x_1,x_2) = \sin(x_1,x_2)$. Let $P_i(x_1,x_2) = \frac{\partial G_i}{\partial x_i}$, $i = 1,2$.

S t e p 1. Choose some $(x_1^{(o)}, x_2^{(o)})$ to be the initial estimate of the solution. Then at some k-th iteration we compute

$$\text{S t e p 2.} \quad W_1^{(k)} = \frac{G_1\left(G_1^{(k-1)}, x_2^{(k-1)}\right) - G_1^{(k-1)}}{1 - P_1\left(G_1^{(k-1)}, x_2^{(k-1)}\right)},$$

$$W_2^{(k)} = \frac{G_2\left(x_1^{(k-1)}, G_2^{(k-1)}\right) - G_2^{(k-1)}}{1 - P_2\left(x_1^{(k-1)}, G_2^{(k-1)}\right)},$$

where $G_i^{(k-1)} = G_i\left(x_1^{(k-1)}, x_2^{(k-1)}\right)$, $i = 1,2$.

Now compute

$$\text{S t e p 3.} \quad x_i^{(k)} = W_i^{(k)} + G_i^{(k-1)}, \quad i = 1,2.$$

S t e p 4. If $\max_i \left| W_i^{(k)} \right| \leqslant 10^{-4}$, the last computed values of $x_i^{(k)}$ will give the solutions. If this inequality is not satisfied, the steps 2 - 4 will be repeated.

This procedure may now be extended to any large system. If $P_i = 0$ for some i, the value of the perturbation parameter will be zero, and the effectiveness of the method will be reduced. If $P_i = 1$ for some i, the method will fail.

For the above system of equations when we started iterations arbitrarily with $x_1^{(o)}$, $x_2^{(o)}$ equal to (999.0, -999.0), (-999.0, 999.0) and (-999.0, -999.0) respectively, the method converged to the solution (0, 0) within 4 iterations.

We will now consider few more applications of this method and compare its effectiveness with that of the other well-known nonlinear functional iterations, namely Picard, Jacobi and Gauss-Seidel iterations.

6. Comparison with other functional iterations

A comparative study between the effectiveness of this method with that of other functional iterations, namely, Picard, Jacobi and Gauss-Seidel iterations will now be studied by virtue of applications. Equations with one, two, and three variables will be considered for solution. The criterion for convergence of iterations in Picard, Jacobi and Gauss-Seidel iterations is given by (12) with $\varepsilon = 10^{-4}$, whereas that for the present scheme is given by (13) with $\varepsilon = 10^{-4}$.

(a) E q u a t i o n s w i t h o n e v a r i a b l e

In this case, the present scheme will be compared with Picard's iterations.

E x a m p l e 1. Solve: $x + \ln x = 0$. Choose $x^{(0)} = 0.5$ and $G(x) = \exp(-x)$. Picard's scheme took 14 iterations and the present scheme took 3 iterations to converge to the solution $x = 0.56712$.

E x a m p l e 2. Solve: $x^5 - 3.7x^4 + 7.4x^3 - 10.8x^2 + 10.8x - 6.8 = 0$. Choose $x^{(0)} = 0.05$ and $G(x) = (6.8 + 10.8x^2 - 7.4x^3 + 3.7x^4 - x^5)/10.8$. Picard's scheme took 14 iterations and the present scheme took 5 iterations to converge to the solution $x = 1.7$.

E x a m p l e 3. Solve: $x - 2.9 \tan x = 0$. Choose $x^{(0)} = -999.0$ and $G(x) = 2.9 \tan x$. Picard's scheme did not converge (even when $x^{(0)} = 0.05$), whereas, the present scheme converged to the solution $x = 0.0$ after 3 iterations.

(b) E q u a t i o n s w i t h t w o v a r i a b l e s

E x a m p l e 4. Solve: $x + y^2 = \sin(xy)$ and $x^2 + y + 1 = \cos(xy^2)$. Choose $x^{(0)} = y^{(0)} = 0.1$; and $G_1(x,y) = \sin(xy) - y^2$, $G_2(x,y) = \cos(xy^2) - 1 - x^2$. Nonlinear Jacobi scheme took 4, nonlinear Gauss-Seidel took 3 iterations and the present method took 4 iterations to converge to the solution $x = y = 0.0$.

E x a m p l e 5. Solve: $x = 0.8542 \cos x + 0.7194 \sin y$ and $y = 0.9764 \sin x + 0.4597 \cos y$. Choose $(x^{(o)}, y^{(o)}) = (1.0, 0.0)$. Nonlinear Jacobi failed, nonlinear Gauss-Seidel took 13 and the present scheme took 9 iterations to converge to the solution $x = 1.0534$ and $y = 1.0695$.

(c) E q u a t i o n s w i t h t h r e e v a r i a b l e s

E x a m p l e 6. Solve: $x - \sin(xyz) = 0$, $y - \cos(xyz) = 0$ and $z - \tan(xyz) = 0$. Choose $x^{(o)} = y^{(o)} = z^{(o)} = 2.0$; and $G_1(x,y,z) = \sin(xyz)$, $G_2(x,y,z) = \cos(xyz)$ and $G_3(x,y,z) = \tan(xyz)$. Nonlinear Jacobi iterations failed. Nonlinear Gauss-Seidel took 5 and the present scheme took 3 iterations to converge to the solution $x = 0.0$, $y = 1.0$ and $z = 0.0$.

E x a m p l e 7. Solve: $x = \tan(xyz)$, $y = \tan(2x + 2y + 2z)$ and $z = \tan x \tan y \tan z$. With $x^{(o)} = y^{(o)} = z^{(o)} = 1.0$, both nonlinear Jacobi and Gauss-Seidel iterations failed to converge. The present scheme converged to the solution $x = y = z = 0$ after 4 iterations. Even with $x^{(o)} = y^{(o)} = z^{(o)} = 0.0001$, nonlinear Jacobi and Gauss-Seidel did not converge within 1000 iterations.

7. Discussions

Like any other numerical method, the present method has its limitations. It is applicable only to systems of nonlinear equations. For a linear system,

$$W_i^{(k)} = 0 \quad \text{for all} \quad i = 1,2,\ldots,n \quad \text{and} \quad k = 1,2,\ldots$$

Hence this scheme reduces to linear Jacobi iterations. Also by computer experimentations it was found that, whenever for a particular system $\frac{\partial G_i}{\partial x_i} = 0$, $W_i = 0$ and the rate of convergence slows down.

The method is found to be not quite effective to solve systems of equations having multiple roots. For example: the system of equations $x = \sin x \cos y$ and $y = 1.5708 \cos x \sin y$ has solutions which are: (0, 0), (0, 1.5708) and (0, -1.5708). When $(x^{(o)}, y^{(o)})$ were chosen as follows: (0.1, -2.0),

(0.1, 0.1) and (0.5, 2.1), the method converged to the solutions (0, -1.5708), (0, 0) and (0, 1.5708) respectively. However, when the initial estimates were arbitrarily chosen as (999.0, 999.0), (333.9, -444.9) and (24.5, -45.9) it converged to (0, 0), (0, 1.5708) and (0, -1.5708) respectively. Indeed it was found consistently that whenever good initial estimates of a particular solution are chosen, the method invariably converged to that solution within few iterations; and if the initial estimates are chosen at random, it usually converges to one of the multiple roots. The cause for this has not yet been found mathematically.

For systems of equations with unique solutions, it was verified computationally that the method exhibits a global convergence property. In the example 3 (of the previous section), when the initial estimates were arbitrarily chosen as -50.0, 999.0, and 85.0 respectively, it converged to the solution $x = 0.0$ within 5 iterations. In the example 5, it converged to the solution $x = 1.0534$ and $y = 1.0695$ within 10 iterations when the initial estimates $(x^{(o)}, y^{(o)})$ were arbitrarily chosen as: (-1.02, 2.0), (5.24, 9.14), (0.01, 8.42), (-3.05, -9.21) and (501.2, 32.05). In the example 7, with arbitrary values assigned to the initial estimates $(x^{(o)}, y^{(o)}, z^{(o)})$ as follows: (0.1, 0.1, 0.1), (-555.8, 897.0, -876.9), (999.0, -999.0, -999.0), (-999.0, 999.0, 999.0), (-999.0, -999.0, -999.0); (999.0, -999.0, 999.0) the present scheme converged systematically to the solution (0.0, 0.0, 0.0) within 10 iterations. It is needless to say that both nonlinear Jacobi and Gauss-Seidel iterations failed.

In several applications it was found that while Jacobi iterations failed to converge, the new scheme converged to the solution within few iterations. We will now study the probable cause for this. Let

$$b_{ii}^{(k-1)} = \frac{1}{1 - p_{ii}^{(k-1)}},$$

where

$$(14) \quad p_{ii}^{(k-1)} = \left(\frac{\partial G_i}{\partial x_i}\right)_{x_1^{(k-1)},\ldots,x_{i-1}^{(k-1)},\ G_i^{(k-1)},\ x_{i+1}^{(k-1)},\ldots,x_n^{(k-1)}}.$$

Then from (6) and (8),

$$x_i^{(k)} = \left(1 - b_{ii}^{(k-1)}\right) G_i^{(k-1)} +$$

$$+ b_{ii}^{(k-1)}\, G_i\left(x_1^{(k-1)},\ldots,x_{i-1}^{(k-1)},\ G_i^{(k-1)},\ x_{i+1}^{(k-1)},\ldots,x_n^{(k-1)}\right),$$

where, $G_i^{(k-1)} = G_i\left(x_1^{(k-1)},\ldots,x_n^{(k-1)}\right)$.

If $b_{ii}^{(k-1)}$ is replaced by zero, the present scheme reduces to nonlinear Jacobi iterations. Thus it is clear that the sequence of iterates generated by the present scheme are different from those given by Jacobi iterations. However, it can be proved that if both techniques generate the sequence of iterates $\{x^{(k)}\}$ such that $x^{(k)} \in D$ for all k and the mapping G is contractive in D satisfying (10), then Jacobi iterations will converge to x^*, and if furthermore, the perturbation vector $W^{(k)}$ satisfies (11), the present method will also converge to the solution x^*. But in general, nonlinear Jacobi iterations will generate iterates $\{x^{(k)}\} \in D_o$, where $D_o \subset R^n$ and $D_o \neq D$. Thus both techniques may not simultaneously converge to the solution. This has been verified in the examples 3, 5, 6 and 7 of section 5. Although no example was found yet, where present scheme of iterations diverged and the nonlinear Jacobi scheme converged to the solution, such a possibility cannot be ruled out. A large number of applications studied so far have indicated that whenever both the present scheme and nonlinear Jacobi iterations converged simultaneously, the former converged faster than the latter in general. The possible cause for this will now be the topic of our discussion.

First, we will assume that the method is converging to the solution so that,

$$\frac{|x_i^{(k)} - x_i^*|}{|x_i^{(k-1)} - x_i^*|} < 1 \quad \text{for all} \quad i.$$

Since we did not assume the existence of $\frac{\partial G_i}{\partial x_j}$ for $i \neq j$, let us assume that for all $x^{(k)} \in D$,

$$(15) \quad G_i\left(x_1^{(k-1)},\ldots,x_{i-1}^{(k-1)},G_i^{(k-1)},x_{i+1}^{(k-1)},\ldots,x_n^{(k-1)}\right) - G_i(x_1^* \ldots x_n^*) =$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{n} q_{ij}^{(k-1)}\left(x_j^{(k-1)} - x_j^*\right) + p_{ii}^{(k-1)}\left(G_i^{(k-1)} - x_i^*\right) +$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 G_i}{\partial x_i^2}\right)_{x_1^{(k-1)},\ldots,G_i^{(k-1)} - \theta\left(G_i^{(k-1)} - x_i^*\right),\ldots,x_n^{(k-1)}} \left(G_i^{(k-1)} - x_i^*\right)^2,$$

where $0 < \theta < 1$, $p_{ii}^{(k-1)}$ is given by (14) and $q_{ij}^{(k-1)}$ are real numbers depending on G_i and x. It may be observed that $q_{ij}^{(k-1)}$ may be chosen such that (15) will represent the contractive property of G.

Now, from (6) we get

$$(16) \quad x_i^{(k)} - x_i^* =$$

$$= W_i^{(k)} + G_i\left(x_1^{(k-1)},\ldots,x_n^{(k-1)}\right) - x_i^*, \quad i=1,2,\ldots,n, \quad k = 1,2,\ldots$$

Recalling that $x^* = G(x^*,\ldots,x^*)$ and applying the contractive property of G as given by the inequality (10) we get,

$$(17) \quad \left|x_i^{(k)} - x_i^*\right| \leqslant \left|W_i^{(k)}\right| + \sum_{j=1}^{n} a_{ij}\left|x_j^{(k-1)} - x_j^*\right|,$$

where a_{ij} are the elements of A.

Now if we replace $W_i^{(k)}$ in (16) by applying (8) and make use of the equation (15), we may get after some simplifications

$$(18)\quad \left|x_i^{(k)} - x_i^*\right| \leqslant \frac{t_i}{2}\left(G_i^{(k-1)} - x_i^*\right)^2 + \sum_{j=1}^{n} \left|q_{ij}^{(k-1)}\right| \left|x_j^{(k-1)} - x_i^*\right|,$$

where

$$\left|q_{ii}^{(k-1)}\right| \geqslant \left|p_{ii}^{(k-1)}\right|,$$

and

$$t_i = \max_k \left| \left(\frac{\partial^2 G_i}{\partial x_i^2}\right)_{x_1^{(k-1)},\ldots,G_i^{(k-1)} - \theta\left(G_i^{(k-1)} - x_i^*\right),\ldots,x_n^{(k-1)}} \right|.$$

Evidently $\left|q_{ij}^{(k-1)}\right|$ are elements of an isotone matrix for all k. Without any loss of generality let us define

$$\max_k \left|q_{ij}^{(k-1)}\right| = a_{ij}, \quad i,j = 1,2,\ldots,n.$$

This indeed is in conformity with the contractive property of G. Hence from (18) we get

$$(19)\quad \left|x_i^{(k)} - x_i^*\right| \leqslant \frac{t_i}{2}\left(G_i^{(k-1)} - x_i^*\right)^2 + \sum_{j=1}^{n} a_{ij}\left|x_j^{(k-1)} - x_j^*\right|.$$

The inequalities (17) and (19) are both derived from the same equation (16). Only contractive property of the operator G was used in (17), whereas the value of $W_i^{(k)}$ as well as contractive property of G were used in (19). Thus recalling the assumption (7b) (which shows that the elements t_i are bounded for all $i = 1,2,\ldots,n$) and the assumption imposed on $W_i^{(k)}$ (that they are small quantities whose squares may be neglected) we may conclude that, near the solution (where $\left|G_i^{(k-1)} - x_i^*\right| < 1$) $\left|W_i^{(k)}\right|$ is proportional to $\left(G_i^{(k-1)} - x_i\right)^2$ for $i = 1,2,\ldots,n$. Since in the present scheme, whenever

convergence takes place, $x^{(k)}$ approaches x^* as fast as $|W^{(k)}|$ approaches zero, the rate of convergence is proportional to $(G_i^{(k-1)} - x_i^*)^2$ for $i = 1,2,\ldots,n$. This was systematically verified by computer experimentations. In the absence of perturbations $G_i^{(k-1)}$ are the nonlinear Jacobi iterates. However, since $|W_i^{(k)}|$ are small near the solution, under the present scheme, $G_i^{(k-1)}$ will approximate the Jacobi iterates. Thus we may conclude that the rate of convergence of the present method is almost quadratic with respect to that of the nonlinear Jacobi iterations.

8. Conclusion

The method developed in this paper is a form of functional iteration and hence it is simple both in theory and in practice. Although it is obtained by simply adding a perturbation vector to the nonlinear Jacobi iterations, it has demonstrated so far better convergence properties with respect to other existing functional iterations. A large number of applications studied so far indicate that the method is effective.

Finally, works are on progress at present to study both theoretically and computationally the properties of a combined iterative scheme consisting of a successive over/under relaxation technique and perturbed Jacobi/Gauss-Seidel iterations. Theoretically or computationally no conclusive results have yet been obtained. However, by computer experimentations it was found so far that the perturbed Jacobi and the perturbed Gauss-Seidel are almost equally effective and the introduction of an arbitrary over/under relaxation parameter in the perturbed iterations does not improve its convergence properties.

REFERENCES

[1] D.F. D a v i d e n k o: On a new method of numerical solution of systems of nonlinear equations, Dokl. Akad. Nauk SSSR 88 (N.S.), (1953) 601-602.

[2] F. D e i s t, L. S e f o r: Solution of systems of nonlinear equations by parameter variation, Computer J. 10 (1967) 78-82.

[3] F. F r e u d e n s t e i n, B. R o t h: Numerical solution of systems of nonlinear equations, J. Assoc. Comput. Mach. 10 (1963) 550-556.

[4] J.M. O r t e g a, W.C. R h e i n b o l d t: Iterative solution of nonlinear equations in several variables, New York 1970.

DEPARTMENT OF MATHEMATICS, EASTERN ILLINOIS UNIVERSITY, CHARLESTON, ILLINOIS 61920, U.S.A.

Received February 25, 1976.