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(Z, Q)-MACHINES AND (Z, Q)-COMPUTATIONS

Introduction

The notions of a (Z,Q)-machine and a (Z,Q)-computable function are introduced in [6]. The notion of a (Z,Q)-machine is a generalization of the notions of Pawlak's machine [1], of a k-machine [2], of a simple continuous machine [3,4] and of a continuous simple Z-machine [5]. In this paper some basic properties of (Z,Q)-machines and (Z,Q)-computable functions are examined. In the second part dependence of (Z,Q)-computability upon (Z_a,Q)-computability of a function f, $f \in X^Q$, $a \in T_Q$, are investigated.

1. Basic notations and definitions

Let $\mathcal{R}$ be the set of all real numbers, and $\mathcal{R}^+$ - the set of all non-negative real numbers. Let us denote the set of all non-negative integers by $\mathcal{N}_0$ and the empty set by $\emptyset$.

Let n be an arbitrary, but fixed, positive integer. By (t,x) we denote the point $(t,x_2,\dots,x_n)\in\mathcal{R}^n$ if $n \geqslant 2$, and the point $(t)\in\mathcal{R}$ if $n = 1$. A subset Ω_n of $\mathcal{R}^n$ we define as follows: $\Omega_n = \left\{(t,x):\ t\in\mathcal{R}^+ \wedge x\in\mathcal{R}^{n-1}\right\}$ if $n \geqslant 2$ and $\Omega_1 = \mathcal{R}^+$.

U_a denotes the set $\left\{(t,x)\in\Omega_n : (t-a,x)\in U\right\}$ for any $a\in\mathcal{R}^+$ and $\emptyset \neq U\subseteq\Omega_n$. If $S\subseteq\Omega_n$, then by T_S we denote the set $\left\{t\in\mathcal{R}^+ : \exists_{x\in\mathcal{R}^{n-1}} (t,x)\in S\right\}$ for $n\geqslant 2$, and S for $n = 1$.

Denote by Q an arbitrary subset of Ω_n which satisfies the following

(1) $$0 \in T_Q$$

(2) $$T_Q - \{0\} \neq \emptyset$$

(3) $$Q_a \subseteq Q \text{ for any } a \in T_Q.$$

The set Q which satisfies the following condition

$$\forall_{(t,x)\in\Omega_n} \left[(t,x) \in Q \Rightarrow (0,x) \in Q\right]$$

is called a T_Q-steady set. We say that the set T_Q is closed under subtraction if $b-a \in T_Q$ for any $a,b \in T_Q$ such that $a < b$.

Let Z be an arbitrary non-empty subset of Q and X denotes arbitrary but fixed non-empty set. By $\mathcal{A}_{Z,Q}$ we denote a set of all operators such that $\emptyset \neq DA \subseteq X^Z$ and $RA \subseteq X^Q$. Assign to every mapping $f : Z_a \to X$, where $a \in T_Q$, the mapping $f^* : Z \to X$ such that $f^*(t,x) = f(t+a,x)$ for any $(t,x) \in Z$. We shall use the following notations: $f_{Z_a} = (f|Z_a)^*$ and $f_{Z_0} = f_Z$.

D e f i n i t i o n 1. An operator $M \in \mathcal{A}_{Z,Q}$ is called a n-dimensional (Z,Q)-machine iff

(4) $$\forall_{f\in DM} \left[(Mf)|Z = f\right]$$

and

(5) $$\forall_{f\in RM} \forall_{a\in T_Q} \left[f_{Q_a} \in RM\right].$$

The set of all n-dimensional (Z,Q)-machines is denoted by $\mathfrak{M}_{Z,Q}$. If $M \in \mathfrak{M}_{Z,Q}$, then the elements of the set DM are called initial functions of the machine M, and the elements of the set RM - the computations of this machine.

E x a m p l e. Let $n = 1$. For an arbitrary positive integer m and $a \in \mathcal{R}^{+} - \{0\}$ we define the following finite sequences $\{A_k\}$, $\{B_k\}$ of subsets of $\mathcal{R}^{+}$.

$$A_k = \begin{cases} \emptyset, & \text{for } k = 1 \\ \langle ka + \frac{a}{m}; \; ka + \frac{ka}{m}), & \text{for } k = 2,\dots,m \end{cases}$$

$$B_k = (ka + \frac{a}{2m}; \; ka + \frac{a}{m}), \quad \text{for } k = 1,\dots,m.$$

The family of these subsets satisfies the following condition

$$\forall_{1 \leqslant i,j \leqslant m} \; \forall_{s_j \in A_j \cup B_j} \left\{ \left[i+j \leqslant m \Longrightarrow (A_i \cup B_i)_{s_j} \subseteq A_{i+j} \right] \vee \vee \left[i+j > m \Longrightarrow (A_i \cup B_i)_{s_j} \subseteq \langle a(m+1); \; +\infty) \right] \right\}.$$

Hence, the set Q defined as follows

$$Q = \{0\} \cup \bigcup_{k=1}^{2m} S_k \cup \langle a(m+1); \; +\infty),$$

where

$$S_k = \begin{cases} A_{\frac{k+1}{2}} & \text{for } k = 2l-1, \; l = 1,\dots,m \\ \text{an arbitrary finite subset of the set } B_{\frac{k}{2}}, & \text{for } k = 2l, \; l = 1,\dots,m. \end{cases}$$

satisfies the conditions (1) - (3).

Thus, the (Z,Q)-machine we may mean as an abstract model of a hybrid machine.

2. Some properties of n-dimensional (Z,Q)-machine

At the beginning we shall quote some results from [6].

L e m m a 1. [6] Any machine $M \in \mathfrak{M}_{Z,Q}$ is a bijection.

Theorem 2. [6] If $M \in \mathfrak{M}_{Z,Q}$, then $f_{Z_a} \in DM$ and $M(f_{Z_a}) = f_{Q_a}$ for any $f \in RM$, $a \in T_Q$.

Lemma 3. [6] If $M \in \mathfrak{M}_{Z,Q}$, $f,h \in RM$, $a,b \in T_Q$ and $f_{Z_a} = h_{Z_b}$, then $f_{Q_a} = h_{Q_b}$.

Definition 2. Let Z be an arbitrary non-empty subset of Q. By a diametr of the set T_Z in the set T_Q (in symbols $|Z|_T$) we mean a number

$$|Z|_T = \sup_{t \in T_Z} t - \inf_{t \in T_Z} t .$$

Lemma 4. If $M \in \mathfrak{M}_{Z,Q}$ and $|Z|_T = \omega \in T_Q$, then $f_{Q_t} = (Mf_{Z_{t-\omega}})_{Q_\omega}$ for any $f \in RM$ and for any $t \in \mathcal{R}^+$ such that $t-\omega \in T_Q$.

Proof. Observe, that for any $t \in \mathcal{R}^+$ such that $t-\omega \in T_Q$ we have $t = \omega + (t-\omega) \in T_Q$. Let (t_o,x_o) be an arbitrary point of Q. Since $t_o \in T_Q$, then $t_o+\omega \in T_Q$, $t_o+t \in T_Q$. By Theorem 2 we get

$$(Mf_{Z_{t-\omega}})_{Q_\omega}(t_o,x_o) = f_{Q_{t-\omega}}(t_o+\omega,x_o) = f(t+t_o,x_o) = f_{Q_t}(t_o,x_o)$$

proving that $(Mf_{Z_{t-\omega}})_{Q_\omega} = f_{Q_t}$.

Lemma 5. If Q is a T_Q-steady set, $|Z|_T = \omega \in T_Q$ and $M \in \mathfrak{M}_{Z,Q}$, then the value $f(t_o,x)$ is uniquely determined by values of function f within set $Z_{t-\omega}$ for any $f \in RM$ and for any point $(t_o,x) \in Q$ such that $t_o-\omega \in T_Q$.

Proof. Let us choose (t_o,x_o) such that $t_o-\omega \in T_Q$. Note that $(0,x) \in Q$ and $(\omega,x) \in Q$. By Lemma 4 we obtain

$$f(t_o,x) = f_{Q_{t_o}}(0,x) = (Mf_{Z_{t_o-\omega}})_{Q_\omega}(0,x) = (Mf_{Z_{t_o-\omega}})(\omega,x)$$

this completes the proof.

E x a m p l e 2. Let $n = 1$, $X = \mathcal{R}^+$, $Q=\{0,1\} \cup \langle 2;+\infty)$, $Z = \{0,1\}$ and set $f(t) = t$ for any $t \in T_Q$. An operator M defined by the formula

$$DM = \left\{ f_{Z_a} : a \in T_Q \right\}, \quad M(f_{Z_a}) \underset{df}{=} f_{Q_a}$$

is (Z,Q)-machine. Since $|Z|_T = \omega = 1$, then for $t = \frac{5}{2}$ we get $t-\omega = \frac{3}{2}$, so $t-\omega \notin T_Q$. Moreover $f(\frac{3}{2}) = \frac{3}{2}$ and $f(\frac{3}{2}) \notin R_{f_{Q_t}}$ for any $t \in T_Q$, $t < \frac{3}{2}$.

T h e o r e m 6. For any machines $M_1, M_2 \in \mathfrak{M}_{Z,Q}$ the following conditions are equivallent

$$\forall_{f \in DM_1 \cap DM_2} \left[M_1(f) = M_2(f) \right] \tag{6}$$

$$\forall_{f,g \in RM_1 \cup RM_2} \left[f_Z = g_Z \Longrightarrow f = g \right]. \tag{7}$$

P r o o f. Let $M_1, M_2 \in \mathfrak{M}_{Z,Q}$ and assume Condition (6) is satisfied. Let $f,g \in RM_1 \cup RM_2$ and $f_Z = g_Z$. If both f and g belong to the same set RM_1 or RM_2, they must be equal, since they are computations of some machine. Suppose now, $f \in RM_1 - RM_2$ and $g \in RM_2 - RM_1$ and $f_Z = g_Z$. Then $g_Z \in DM_1 \cap DM_2$ and Condition (6) implies $g = f$, contrary to the equality $(RM_2 - RM_1) \cap (RM_1 - RM_2) = \emptyset$.

For the converse, let (7) holds for M_1, M_2 and let $f \in DM_1 \cap DM_2$. There exist $g \in RM_1$ and $h \in RM_2$ such that $g_Z = f = h_Z$. Consequently, by (7), $g = h$. On the other hand

$$g = M_1(g_Z) = M_1 f = M_2(h_Z) = M_2 f = h$$

proving (6).

3. (Z,Q)-computable functions

D e f i n i t i o n 3. [6] A function $f \in X^Q$ is said to be (Z,Q)-computable iff there exists $M \in \mathfrak{M}_{Z,Q}$ such that $f \in RM$.

T h e o r e m 7. [6] A necessary and sufficient condition for a function $f \in X^Q$ to be (Z,Q)-computable is

$$\forall_{a,b \in T_Q} \left[f_{Z_a} = f_{Z_b} \Rightarrow f_{Q_a} = f_{Q_b} \right] \tag{8}$$

D e f i n i t i o n 4. A function $f \in X^Q$ is said to be Z-injective iff

$$\forall_{a,b \in T_Q} \left[a \neq b \Longrightarrow f_{Z_a} \neq f_{Z_b} \right]. \tag{9}$$

By Definition 4 any Z-injective function $f \in X^Q$ is (Z,Q)-computable.

D e f i n i t i o n 5. A function $f \in X^Q$ is said to be periodic with respect to the variable t (or simply: a periodic function) of period τ, $\tau \in T_Q - \{0\}$ iff

$$\forall_{(t,x) \in Q} \left[f(t+\tau,x) = f(t,x) \right]. \tag{10}$$

L e m m a 8. For any $f \in X^Q$ and any $a,b \in T_Q$ such that $b-a \in T_Q - \{0\}$ the following conditions are equivallent

(11) $f_{Q_a} = f_{Q_b}$,

(12) f_{Q_a} is a periodic function of period $b-a$.

P r o o f. Let $f \in X^Q$, $a,b \in T_Q$, such that $b-a \in T_Q - \{0\}$ and $f_{Q_a} = f_{Q_b}$. Given a point $(t,x) \in Q$ we have

$$f_{Q_a}(t+b-a,x) = f(t+b,x) = f_{Q_b}(t,x) = f_{Q_a}(t,x)$$

proving (12). Conversely, suppose that $a,b \in T_Q$, $b-a \in T_Q-\{o\}$ and f satisfy Condition (12). Then, for any $(t,x) \in Q$ we get

$$f_{Q_a}(t,x) = f_{Q_a}(t+b-a,x) = f(t+b,x) = f_{Q_b}(t,x)$$

verifying that $f_{Q_a} = f_{Q_b}$.

C o r o l l a r y 1. If the set T_Q is closed under subtraction, then for any $f \in X^Q$ and $a,b \in T_Q$, $a < b$, the following conditions are equivallent

(11') $f_{Q_a} = f_{Q_b}$,

(12') f_{Q_a} is a periodic function of period $b-a$.

T h e o r e m 9. If $f \in X^Q$ is (Z,Q)-computable, then exactly one of the following conditions is satisfied

(13) f is Z-injective

(14) there exist $\alpha,\beta \in T_Q$, $\alpha < \beta$, such that

$f_{Z_\alpha} = f_{Z_\beta}$ and for any $a,b \in T_Q$ such that $b-a \in T_Q - \{0\}$, $f_{Z_a} = f_{Z_b}$ implies that f_{Q_a} is a periodic function of period $b-a$.

The proof is obvious by Definition 4, Theorem 7 and Lemma 8.

C o r o l l a r y 2. If the set T_Q is closed under subtraction, then a function $f \in X^Q$ is (Z,Q)-computable iff exactly one of the following conditions is satisfied

(13') f is Z-injective

(14') there exist $\alpha,\beta \in T_Q$, $\alpha < \beta$, such that

$f_{Z_\alpha} = f_{Z_\beta}$ and for any $a,b \in T_Q$ such that $a < b$ $f_{Z_a} = f_{Z_b}$ implies that f_{Q_a} is a periodic function of period $b-a$.

T h e o r e m 10. If $\overline{\overline{X}} \geqslant 2$, then for every set $Z \subsetneqq Q$ there exists a function $f \in X^Q$, which is not (Z,Q)-computable.

P r o o f. Since $\overline{\overline{X}} \geqslant 2$, there exist $y_1, y_2 \in X$ such that $y_1 \neq y_2$. Let $(t_0, x_0) \in Q - Z$. If $t_0 \neq 0$, let a function f be defined as follows

$$f(t,x) = \begin{cases} y_1 & \text{for } (t,x) \in Q - \{(t_0,x_0),(2t_0,x_0)\} \\ y_2 & \text{for } (t,x) \in \{(t_0,x_0),(2t_0,x_0)\}. \end{cases}$$

It can be easy verified that $f_{Z_{t_0}} = f_{Z_{2t_0}}$. On the other hand

$$f_{Q_{t_0}}(t_0,x_0) = f(2t_0,x_0) = y_2 \neq y_1 = f(3t_0,x_0) = f_{Q_{2t_0}}(t_0,x_0)$$

and $f_{Q_{t_0}} \neq f_{Q_{2t_0}}$, proving that f is not (Z,Q)-computable. For $t_0 = 0$ a function f is defined by the following formula

$$f(t,x) = \begin{cases} y_1 & \text{for } (t,x) = (0,x_0) \\ y_2 & \text{for } (t,x) \in Q - \{(0,x_0)\}. \end{cases}$$

Let $a \in T_Q - \{0\}$. Thus, since $f_Z = f_{Z_a}$ and $f \neq f_{Q_a}$, f is not (Z,Q)-computable.

T h e o r e m 11. If $\overline{\overline{X}} \geqslant 2$, then there exists a function $f \in X^Q$ which is not (Z,Q)-computable for any set

$$Z \in \mathcal{Z} = \{Z : Z = (\langle 0; \omega) \times \mathcal{R}^{n-1}) \cap Q \wedge \omega \in \mathcal{R}^+\}.$$

P r o o f. Let $y_1, y_2 \in X$ such that $y_1 \neq y_2$, $a \in T_Q - \{0\}$ and $(a,x_0),(0,x_0) \in Q$. Consider the function

$$f(t,x) = \begin{cases} y_1 & \text{for } (t,x) = (k^2a,x_0)\ k \in \mathcal{N} \\ y_2 & \text{for } (t,x) \in Q - \bigcup_{k \in \mathcal{N}} \{(k^2a,x_0)\}. \end{cases}$$

Let $Z \in \mathcal{Z}$ and $\omega \in \mathcal{R}^+$ such that $Z = (\{0;\omega) \times \mathcal{R}^{n-1}) \cap Q$. We must check two cases

1) $0 \leqslant \omega < a$. Then $f_Z = f_{Z_{2a}}$ and $f \neq f_{Q_{2a}}$.

2) $\omega \geqslant a$. There exists a natural number n such that $n^2 a \leqslant \omega < (n+1)^2 a$. Then we conclude

$$f_{Z_{k_2^2 a}} = f_{Z_{k_1^2 a}} \quad \text{and} \quad f_{Q_{k_2^2 a}} \neq f_{Q_{k_1^2 a}}, \quad \text{where} \quad k_1, k_2 \geqslant \frac{n(n+2)}{2},\ k_1 \neq k_2.$$

It follows, from (1) - (2), that f is not (Z,Q)-computable.

L e m m a 12. [6] If $f \in X^Q$ is (Z,Q)-computable, then for every set G such that $Z \subseteq G \subseteq Q$, f is (G,Q)-computable. These last two results have an important corollary

C o r o l l a r y 3. If $\overline{\overline{X}} \geqslant 2$, then there exists a function $f \in X^Q$ which is not (Z,Q)-computable for any $Z \subseteq Q$ such that $|Z|_T < +\infty$.

In the next part we shall give a few results concerning dependence of (Z,Q)-computability upon (Z_a,Q)-computability of function $f \in X^Q$, $a \in T_Q - \{0\}$.

T h e o r e m 13. If T_Q is closed under subtraction, then every (Z,Q)-computable function of period $\tau, \tau > 0$, is (Z_a,Q)-computable for any $a \in T_Q$.

P r o o f. Let f be a (Z,Q)-computable periodic function of period $\tau, \tau > 0$ and $a \in T_Q - \{0\}$. There exists a natural number k such that $k\tau > a$ and by assumptions, $k\tau - a \in T_Q$. Suppose $f_{(Z_a)_b} = f_{(Z_a)_c}$ for certain $b, c \in T_Q$. This implies $f_{Q_{a+b}} = f_{Q_{a+c}}$, since f is (Z,Q)-computable. Let $(t,x) \in Q$, then $t+k\tau-a \in T_Q$ and we obtain

$$f_{Q_b}(t,x) = f(t+b,x) = f(t+k\tau-a+b+a,x) =$$

$$= f_{Q_{a+b}}(t+k\tau-a,x) = f_{Q_{a+c}}(t+k\tau-a,x) = f(t+c,x) = f_{Q_c}(t,x).$$

This completes the proof, provided that $f_{Q_b} = f_{Q_c}$.

It is easy to show that not every (Z,Q)-computable function is (Z_a,Q)-computable for $a \in T_Q - \{0\}$.

T h e o r e m 14. If $f \in X^Q$ is Z-injective, then it is Z_a-injective for every $a \in T_Q$.

The proof is immediate from Definition 4 and Theorem 7.

Of course, if $f \in X^Q$ is Z-injective, then f is (Z_a,Q)-computable for every $a \in T_Q$.

T h e o r e m 15. If T_Q is closed under subtraction and $f \in X^Q$ is (Z,Q)-computable and Z_a-injective for certain $a \in T_Q$, then f is Z-injective.

P r o o f. Assume that f is Z_a-injective for certain $a \in T_Q$. Thus f_{Q_a} is Z-injective. Suppose that $f_{Z_b} = f_{Z_c}$ for $b,c \in T_Q$, $b < c$. By Corollary 3 f_{Q_b} is a periodic function of period c-b. It is impossible, since the formula $\left[f_{(Q_a)_{b-a}} = f_{Q_b}\right] \vee \left[f_{(Q_b)_{a-b}} = f_{Q_a}\right]$ is true for any $b,a \in T_Q$. Consequently, $f_{Z_b} \neq f_{Z_c}$ for $b \neq c$.

E x a m p l e 3. Let $n = 1$, $\overline{\overline{X}} \geqslant 2$, $Z = \langle 0;\tau) \cap Q$, $\tau \in T_Q - \{0\}$ and $T_Z - \{0\} \neq \emptyset$. Let $y_1, y_2 \in X$ such that $y_1 \neq y_2$. We define a function f as follows

$$f(t) = \begin{cases} y_1 & \text{for } n\tau \leqslant t < (n + \frac{1}{n-1})\tau,\ t \in T_Q,\ n \in \mathcal{N} - \{1\} \\ y_2 & \text{for } 0 \leqslant t < 2\tau \vee (n + \frac{1}{n-1})\tau \leqslant t < (n+1)\tau, \\ & \qquad t \in Q,\ n \in \mathcal{N} - \{1\} \end{cases}$$

f is $Z_{2\tau}$-injective. It is not (Z,Q)-computable, since $f_Z = f_{Z_\tau}$ and $f \neq f_{Q_\tau}$.

This example shows that f may be Z-injective for certain $a \in T_Q - \{0\}$ and not (Z,Q)-computable at the same time.

In the last part we shall give few properties of (Z,Q)--computable functions which are generalizations of adequate results obtained previously for Z-computable functions [5].

T h e o r e m 16. If $f \in X^Q$, $(t_0,x_0) \in Z$ and $\overline{\overline{X}} \geqslant \mathcal{L}$, then there exists a (Z,Q)-computable function g such that $g(t,x) = f(t,x)$ for any $(t,x) \in Q - A_{(t_0,x_0)}$, where

$$A_{(t_0,x_0)} = \{(t,x) \in Q : x = x_0 \wedge t \in \langle t_0; +\infty) \cap T_Q\}.$$

P r o o f. It follows, from the definition of $A_{(t_0,x_0)}$, that $\overline{\overline{A_{(t_0,x_0)}}} = \overline{\overline{T_{A(t_0,x_0)}}}$ and $\overline{\overline{A_{(t_0,x_0)}}} \leqslant \mathcal{L}$. Then there exists a subset $Y \subseteq X$ such that $\overline{\overline{A_{(t_0,x_0)}}} = \overline{\overline{Y}}$. Consequently, there exists an one-to-one mapping $h : A_{(t_0,x_0)} \longrightarrow Y$. Define a function g as follows

$$g(t,x) = \begin{cases} f(t,x) & \text{for} \quad (t,x) \in Q - A_{(t_0,x_0)} \\ h(t,x) & \text{for} \quad (t,x) \in A_{(t_0,x_0)}, \end{cases}$$

g is Z-injective, so it is (Z,Q)-computable.

C o r o l l a r y 4. If $f \in X^Q$ and $\overline{\overline{X}} \geqslant \mathcal{L}$, then for every $Z \subseteq Q$ there exists a (Z,Q)-computable function g, whose values differ from values of f at most in a set $A_{(t_0,x_0)}$, where (t_0,x_0) is an arbitrary point of Z.

T h e o r e m 17. If X is a metric space, $(t_0,x_0) \in Z$, $f \in X^Q$ is continuous on a set $A_{(t_0,x_0)}$ and $\overline{\overline{X}} \geqslant \mathcal{L}$, then there exists a (Z,Q)-computable function g such that

$$f(t,x) = g(t,x) \quad \text{for any} \quad (t,x) \in Q - A_{(t_0,x_0)}$$

and

$$f(t,x) = \lim_{(t,u)\to(t,x)} g(t,u) \quad \text{for any} \quad (t,x) \in A_{(t_0,x_0)}.$$

P r o o f. Denote by g - a function defined in the proof of Theorem 16. It is (Z,Q)-computable. Sice f is continuous on $A_{(t_0,x_0)}$, then for any $(t,x) \in A_{(t_0,x_0)}$ we have

$$f(t,x) = \lim_{(t,u)\to(t,x)} f(t,u) = \lim_{(t,u)\to(t,x)} g(t,u)$$

and

$$f(t,x) = g(t,x) \quad \text{for} \quad (t,x) \in Q - A_{(t_0,x_0)}.$$

C o r o l l a r y 5. If X is a metric space, $f \in X^Q$ is a function continuous on Q, $n \geqslant 2$ and $\overline{\overline{X}} \geqslant \mathcal{L}$, then for every $Z \subseteq Q$ there exists a (Z,Q)-computable function g such that

$$\lim_{(t,u)\to(t,x)} g(t,u) = f(t,x) \quad \text{for} \quad (t,x) \in Q.$$

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