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A TWO-SIDED OPERATIONAL CALCULUS

In this note we describe an operational algebra with a family of scalar products, in respect to which the derivatives and the integrals of Dirac's delta distribution $\delta(t)$ form an orthogonal basis. The operational calculus is founded on the two-sided convolution product

$$g(t) * f(t) = \int_{-\infty}^{+\infty} g(t-x)f(x)dx, \tag{1}$$

but the well known zero divisors (cf. J. Mikusiński [5]) are excluded here. A special two-sided operational calculus has been already considered by Á. Száz [7] and G. Krabbe [8]. However, the limits of integration of the convolution product (1) considered in [7] and [8] are not $-\infty$ and $+\infty$, but 0 and t, respectively.

1. The basic algebras

We start with the complex vector space A of all Lebesgue integrable functions $f(t)$ for which the two-sided Laplace transform

$$F(p) = L\{f(t)\} = \int_{-\infty}^{+\infty} e^{-pt} f(t)dt \tag{2}$$

is absolutely convergent in a certain strip $0 < \operatorname{Re} p < \sigma$ with a number σ dependent on $f(t)$, and for which $F(p)$ is holomorphic for $0 < |p| < \sigma$. The equality in A is to be understood as equality almost everywhere. The space A with the product (1) is in fact an algebra. We denote by B the image of A by the transform L. The space B with the usual operations is also an algebra. Every element of B possesses, for $0 < |p| < \sigma$, a Laurent expansion

$$F(p) = \sum_{n=-\infty}^{+\infty} f_n p^n \tag{3}$$

with

$$f_n = \frac{1}{2\pi i} \int_{|p|=\sigma/2} F(p) p^{-n-1} dp \ ,$$

where the integral is taken in the positive direction.

It is well known from the theory of Laplace transform (cf. [2], [4], [6]) that L is an isomorphism between A and B, i.e. L is linear and satisfies the relation

$$L\{g(t) * f(t)\} = G(p)F(p) \tag{4}$$

for arbitrary elements $f(t), g(t) \in A$ with $G(p) = L\{g(t)\}$, and the inverse transform L^{-1} exists. Furthermore, L has the following properties

$$L\left\{\int_{-\infty}^{t} f(x)dx\right\} = \frac{1}{p} F(p), \tag{5}$$

$$L\{f(t+\lambda)\} = e^{\lambda p} F(p), \tag{6}$$

$$L\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right), \tag{7}$$

$$L\{tf(t)\} = -F'(p), \tag{8}$$

where λ, α are real constants with $\alpha > 0$. In particular, the elements in the brackets on the left-hand sides of these formulas always belong to A. The algebra A contains all integrable functions $f(t)$ with $f(t) \equiv 0$ for $|t| > T$ with a certain T dependent on $f(t)$, as well as it contains functions of the kind $e^{-\alpha t^2}$, $e^{-\alpha|t|}$ with $\mathrm{Re}\,\alpha > 0$. Another element of A is Heaviside's jump function

$$h(t) = \begin{cases} 1 & \text{for} \quad t \geqslant 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Using the notations

$$f_+(t) = f(t)h(t), \; f_-(t) = f(t) - f_+(t),$$

we state that the functions t_+^n, $e_+^{-\alpha t}$, $e_-^{\beta t}$ with integers $n \geqslant 0$ and $\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}\beta > 0$ belong to A and that

$$L\{t_+^n\} = n!p^{-n-1}, \; L\{e_+^{-\alpha t}\} = (p+\alpha)^{-1}, \; L\{e_-^{\beta t}\} = -(p-\beta)^{-1}.$$

Every element $F(p) \in B$ has the decomposition $F(p) = F_1(p) + F_2(p)$ with

$$(9) \qquad F_1(p) = \sum_{n=1}^{\infty} f_{-n}p^{-n}, \quad F_2(p) = \sum_{n=o}^{\infty} f_n p^n.$$

Let $f_\nu(t) = L^{-1}\{F_\nu(p)\}$ for $\nu = 1,2$ and $f_o(t) = \sum_{n=o}^{\infty} \frac{1}{n!} f_{-n-1} t^n$. Hence we have $f_1(t) = f_o(t)h(t)$ and $f_o(t) = O(e^{\delta|t|})$ for all fixed $\delta > 0$ and all real t. We also use analogous notations for other functions.

L e m m a 1. If $F(p), G(p) \in B$, then choosing $|p| \leqslant \varepsilon$ with a sufficiently small ε, we have the equation

$$L^{-1}\left\{\frac{1}{2\pi i}\int_{|z|=2\varepsilon} G(p-z)F(z)dz\right\} = g_2(t)f_o(t) - g_o(t)f_2(t).$$

P r o o f. Replacing $G(p-z)$ and $F(z)$ by their Laurent expansions and considering the formula

$$\frac{1}{2\pi i}\int_{|z|=2\varepsilon} (p-z)^m z^n dz = 0$$

for $m,n \geqslant 0$ as well as for $m,n < 0$, we obtain the equations

$$\frac{1}{2\pi i}\int_{|z|=2\varepsilon} G(p-z)F(z)dz =$$

$$= \frac{1}{2\pi i}\int_{|z|=2\varepsilon}\sum_{n=0}^{\infty}\Big(G_2(p-z)f_{-n-1}z^{-n-1} + g_{-n-1}(p-z)^{-n-1}F_2(z)\Big)dz =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(f_{-n-1}G_2^{(n)}(p) - g_{-n-1}F_2^{(n)}(p)\Big) =$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\,L\Big\{f_{-n-1}t^n g_2(t) - g_{-n-1}t^n f_2(t)\Big\}.$$

Since the series for $f_0(t)$ and $g_0(t)$ have majorants of rank $O(e^{\delta|t|})$ with an arbitrary $\delta > 0$, we can change the order of summation and the Laplace transform for $\delta < \mathrm{Re}\, p \leqslant \varepsilon < \sigma - \delta$; hence the lemma is proved.

Now for sufficiently small number ε we introduce in B the family of the well known scalar product

(10) $$(F(p),G(p))_\varepsilon = \frac{1}{2\pi i}\int_{|p|=\varepsilon} F(p)\overline{G(p)}\,\frac{dp}{p} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} F(\varepsilon e^{i\vartheta})\overline{G(\varepsilon e^{i\vartheta})}\,d\vartheta = \sum_{n=-\infty}^{+\infty} f_n\overline{g_n}\varepsilon^{2n},$$

where g_n are the coefficients of the Laurent expansion of $G(p)$. Hence the isomorphism L generates in A the scalar products

(11) $$(f(t),g(t))_\varepsilon = (F(p),G(p))_\varepsilon$$

and the scalar products generate the norms

$$\|F(p)\|_\varepsilon = \sqrt{(F(p),G(p))_\varepsilon} = \sqrt{(f(t),g(t))_\varepsilon} = \|f(t)\|_\varepsilon.$$

Under additional assumptions there exist explicit representations for the scalar products in A.

L e m m a 2. If the Laplace integral (2) converges absolutely for $|\mathrm{Re}\ p| \leqslant \varepsilon$, we have the representation

$$(f(t),g(t))_\varepsilon = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I_0(2\varepsilon\sqrt{xy})f(x)\overline{g(y)}dxdy, \tag{12}$$

where

$$I_0(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} z^n$$

is a modified Bessel function.

P r o o f. Setting the series for the kernel $I_0(2\varepsilon\sqrt{xy})$ into (12) and changing the order of summation and integration which is allowed since

$$I_0(2\varepsilon\sqrt{|xy|}) \leqslant e^{\varepsilon|x|}\,e^{\varepsilon|y|},$$

we get the former series representation (10) for the scalar product (11). In this series the terms with $n < 0$ vanish in view of the fact that $F(p)$ and $G(p)$ are holomorphic for $p = 0$, whereas for $n \geqslant 0$ we have the expression

$$f_n = \frac{1}{n!} F^{(n)}(0) = \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} f(x)x^n dx$$

and an analogous expression for g_n.

2. The operator algebras

Let A^*, B^* be the inductive limits of A, B, respectively. This means that B^* is the set of all functions $\Phi(p)$ for which there exist a sequence $F_n(p) \in B$ and a number $\varepsilon^* > 0$ with $\|\Phi(p) - F_n(p)\|_\varepsilon \to 0$ for all ε with $0 < \varepsilon \leqslant \varepsilon^*$. The definition of A^* is analogous. The convergence in B^* is equivalent to uniform convergence on every compact set S in a sufficiently small neighbourhood of $p = 0$ with $0 \notin S$.

T h e o r e m 1. The elements of B^* are exactly the functions $\Phi(p)$ which are holomorphic for $0 < |p| < \sigma$ with a certain number σ dependent on $\Phi(p)$.

P r o o f. From the definition of B and the definition of the convergence it follows immediately that all functions in B^* are holomorphic in a neighbourhood of $p = 0$, possibly with the exception of $p = 0$. So we only have to show that all functions of this kind belong to B^*. For this purpose we consider for a fixed integer $m > 0$ the sequence

$$f_n(t) = n^m \sum_{\mu=0}^{m} \binom{m}{\mu} (-1)^\mu h(t - \tfrac{\mu}{n})$$

with the Laplace transforms

$$F_n(p) = \frac{n^m}{p} \left(1 - e^{-\frac{p}{n}}\right)^m .$$

Since $F_n(p) \rightarrow p^{m-1}$ uniformly for $|p| \leqslant 1$, we have $p^m \in B^*$ for all integers m. Consequently every series

$$\Phi(p) = \sum_{n=-\infty}^{+\infty} \varphi_n p^n ,$$

which is convergent for $0 < |p| < \sigma$, i.e. uniformly convergent in every compact subset, represents an element of B^*.

T h e o r e m 2. The operations in the brackets on the left-hand sides of (4) - (8) as well as the differentiation $f'(t)$ with

$$L\left\{f'(t)\right\} = pF(p) ,$$

where $f(t)$ is an absolutely continuous function, can be extended to continuous operations from A^* into A^*.

P r o o f. In view of the isomorphism L which can be extended to an isomorphism between A^* and B^* which is measure preserving with respect to our norms, it suffices to show that the corresponding operations by the extended

Laplace transform are continuous operations from B^* into B^*. But this follows from the equations

$$\|p^{-1}F(p)\|_\varepsilon = \varepsilon^{-1}\|F(p)\|_\varepsilon,$$

$$\|pF(p)\|_\varepsilon = \varepsilon\|F(p)\|_\varepsilon,$$

$$\|F(\tfrac{p}{\alpha})\|_\varepsilon = \|F(p)\|_{\varepsilon/\alpha}$$

and from the inequalities

$$\|e^{\lambda p}F(p)\|_\varepsilon \leqslant e^{|\lambda|\varepsilon}\|F(p)\|_\varepsilon,$$

$$\|G(p)F(p)\|_\varepsilon \leqslant M_\varepsilon\|F(p)\|_\varepsilon,$$

$$\|F'(p)\|_{2\varepsilon} \leqslant \varepsilon^{-1}(\|F(p)\|_\varepsilon + 3\|F(p)\|_{3\varepsilon})$$

with $M_\varepsilon = \max|G(p)|$ for $|p| = \varepsilon$ with sufficiently small ε.

The last estimation follows from Cauchy's formula

$$F'(p) = \frac{1}{2\pi i}\int\limits_{|z|=3\varepsilon}\frac{F(z)dz}{(z-p)^2} - \frac{1}{2\pi i}\int\limits_{|z|=\varepsilon}\frac{F(z)dz}{(z-p)^2}$$

with $|p| = 2\varepsilon$ and

$$\left|\frac{1}{2\pi i}\int\frac{F(z)dz}{(z-p)^2}\right| \leqslant \sqrt{\frac{1}{2\pi}\int|F(z)|^2\,|dz|}\sqrt{\frac{1}{2\pi}\int\frac{|dz|}{|z-p|^4}} \leqslant \frac{k}{\varepsilon}\,\|F(p)\|_{k\varepsilon}$$

with $|z| = k\varepsilon$ for $k = 1$ and 3.

C o r o l l a r y. A^* and B^* are isomorphic algebras.

For the elements of A^* it is customary to use notations like $\varphi(t)$, $\psi(t)$,... and to transfer also the other notations from A to A^* by

$$L^{-1}\left\{\Psi(p)\Phi(p)\right\} = \psi(t)*\varphi(t),$$

$$L^{-1}\left\{p^{-1}\Phi(p)\right\} = \int_{-\infty}^{t}\varphi(x)dx\,,$$

$$L^{-1}\left\{p\Phi(p)\right\} = \varphi'(t)\,,$$

$$L^{-1}\left\{e^{\lambda p}\Phi(p)\right\} = \varphi(t+\lambda),$$

$$L^{-1}\left\{\Phi(\tfrac{p}{\alpha})\right\} = \alpha\varphi(\alpha t)\,,$$

$$L^{-1}\left\{\Phi'(p)\right\} = -t\varphi(t)$$

with $\Phi(p) = L\left\{\varphi(t)\right\}$, $\Psi(p) = L\left\{\psi(t)\right\}$. For the higher derivatives of $\varphi(t)$ we write as usual $\varphi^{(n)}(t)$, and for the iterated integrals we write $\varphi^{(-n)}(t)$ with integers $n \geqslant 0$. Furthermore we use the notation $h'(t) = \delta(t)$, so that we have

$$L\left\{\delta^{(n)}(t)\right\} = p^n$$

for all integers n and $\delta^{(-n-1)}(t) = \frac{t_+^n}{n!}$ for $n \geqslant 0$.

T h e o r e m 3. For all $\varphi(t), \psi(t) \in A^*$ the following developments are valid

$$\varphi(t+\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n\varphi^{(n)}(t), \tag{13}$$

$$\psi(t)*\varphi(t) = \sum_{n=-\infty}^{+\infty} \psi_n\varphi^{(n)}(t), \tag{14}$$

$$\varphi(t) = \sum_{n=-\infty}^{+\infty} \varphi_n\delta^{(n)}(t), \tag{15}$$

φ_n and ψ_n being the coefficients in the Laurent expansions of $\Phi(p)$ and $\Psi(p)$, respectively, as well as the relation

$$\varphi'(t) = \lim_{\lambda\to 0}\left[\frac{1}{\lambda}\left(\varphi(t+\lambda) - \varphi(t)\right)\right].$$

P r o o f. By the definition of convergence it suffices to prove the corresponding assertions in B^*. But in B^* we have, of course,

$$e^{\lambda p}\Phi(p) \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n p^n \Phi(p) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n L\left\{\varphi^{(n)}(t)\right\},$$

$$\Psi(p)\Phi(p) = \sum_{n=-\infty}^{+\infty} \psi_n p^n \Phi(p) = \sum_{n=-\infty}^{+\infty} \psi_n L\left\{\varphi^{(n)}(t)\right\},$$

$$\Phi(p) = \sum_{n=-\infty}^{+\infty} \varphi_n p^n = \sum_{n=-\infty}^{+\infty} \varphi_n L\left\{\delta^{(n)}(t)\right\}$$

and

$$\Phi(p) = \lim_{\lambda\to 0}\left(\frac{1}{\lambda}\,(e^{\lambda p} - 1)\Phi(p)\right).$$

3. Supplementary remarks

1^o. In Theorem 3 the series (13) for $\varphi(t+\lambda)$ is nothing else than Taylor's expansion and, in view of the formula

$$(p^n, p^m)_\varepsilon = (\delta^{(n)}(t), \delta^{(m)}(t))_\varepsilon = 0$$

for $n \neq m$ and the formula

$$(\varphi(t), \delta^{(n)}(t))_\varepsilon = (\Phi(p), p^n)_\varepsilon = \varepsilon^{2n}\varphi_n,$$

the series (15) for $\varphi(t)$ is a Fourier series, i.e. the elements $\delta^{(n)}(t)$ form an orthogonal basis of A^*.

2^o. The convergence in A is different from the usual one. We show this by an example of a sequence the limit of which in the usual sense is not the same as the limit in A. This example is given by the well-known development (cf. e.g. [2])

$$e_+^{-\alpha t} = \frac{\beta}{\alpha} \sum_{n=0}^{\infty} \left(1 - \frac{\beta}{\alpha}\right)^n e_+^{-\beta t} L_n(\beta t) \tag{16}$$

with $|\alpha-\beta| < |\alpha|$, and the Laguerre polynomials

$$L_n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-1)^\nu}{\nu!} x^\nu .$$

In the case $\operatorname{Re}\beta \geqslant 0$ the Laplace transform of the right-hand side of (16) is equal to

$$\frac{\beta}{\alpha}\sum_{n=0}^{\infty} \left(1-\frac{\beta}{\alpha}\right)^n \frac{p^n}{(p+\beta)^{n+1}} = \frac{1}{p+\alpha},$$

where the convergence is uniform for $|p| \leqslant |p+\beta|$, i.e. in particular for $|p| \leqslant \frac{|\beta|}{2}$, if $\beta \neq 0$. But we have

$$L^{-1}\left\{\frac{1}{p+\alpha}\right\} = \begin{cases} e_+^{-\alpha t} & \text{for } \operatorname{Re}\alpha \geqslant 0, \\ -e_-^{-\alpha t} & \text{for } \operatorname{Re}\alpha < 0, \end{cases}$$

so that in the case $\operatorname{Re}\alpha < 0$ the limit depends on the notion of convergence. The reason for this difference is that in the case $\operatorname{Re}\alpha < 0$ the element $e_+^{-\alpha t}$ does not belong to A. Besides, let us remark that for every α with $\operatorname{Re}\alpha < 0$ and $\operatorname{Im}\alpha \neq 0$ exists a β with $|\alpha-\beta| < |\alpha|$ and $\operatorname{Re}\beta \geqslant 0$, e.g. $\beta = i\operatorname{Im}\alpha$.

3^0. For $\Phi(p) \in B^*$ the elements $e^{\lambda p}\Phi(p)$ and $\Phi(\alpha p)$ also belong to B^* in the case of complex numbers λ, α with $\alpha \neq 0$. This gives us a possibility to define $\varphi(t+\lambda)$ and $\varphi(\alpha t)$ for complex "arguments", however, for functions $\varphi(t)$ these elements may differ from $\varphi(t+\lambda)$ and $\varphi(\alpha t)$, respectively, in the ordinary sense. Let $\lambda = i\omega$. We put

$$u(t,\omega) = \frac{1}{2}\,(\varphi(t+i\omega)+\varphi(t-i\omega)),\quad v(t,\omega) = \frac{1}{2i}\,(\varphi(t+i\omega)-\varphi(t-i\omega))$$

with

$$L\{u(t,\omega)\} = \Phi(p)\cos\omega p,\quad L\{v(t,\omega)\} = \Phi(p)\sin\omega p.$$

In the case $\Psi(p,\omega), \frac{\partial}{\partial\omega}\Psi(p,\omega) \in B^*$ we define

$$\psi_\omega(t,\omega) = L^{-1}\left\{\frac{\partial}{\partial\omega}\Psi(p,\omega)\right\}.$$

Then it is easy to see that the Cauchy-Riemann equations

$$u_t(t,\omega) = v_\omega(t,\omega), \; u_\omega(t,\omega) = -v_t(t,\omega)$$

are satisfied.

For the elements $\Phi(p) \in B^*$ with $\varphi_{-1} = 0$ the integrals

$$\int \Phi(p)dp = \int_0^p \Phi_2(z)dz - \int_p^\infty \Phi_1(z)dz$$

(cf. (9)) also belong to B^*. This gives us a possibility to define an operator r for the corresponding elements of A^* by

$$r\varphi(t) = L^{-1}\left\{\int_p^\infty \Phi_1(z)dz - \int_0^p \Phi_2(z)dz\right\}.$$

This operator is a right inverse of the multiplier t, i.e. $tr = 1$, whereas $q = 1 - rt$ is a projection operator (cf. [3]) with

$$q\varphi(t) = \varphi_0\delta(t).$$

4°. It is possible to extend B^* to the algebra B_0 of all functions $\Phi(p)$ which are holomorphic in the points of an interval $0 < p < \sigma$ with a sufficiently small σ dependent on $\Phi(p)$. This algebra is a completion of B, if we use the uniform convergence in every compact subset of a small neighbourhood of the interval $0 < p < \sigma$. In B_0 the following equations are valid

$$\frac{1}{p}\Phi(p) = \lim_{\lambda\to+0}\left(\frac{\lambda}{1-e^{-\lambda p}}\Phi(p)\right) = \lim_{\lambda\to+0}\left(\lambda\sum_{n=0}^{\infty} e^{-n\lambda p}\Phi(p)\right).$$

Hence, in the corresponding algebra A_0 we obtain the limit relation

$$\int_{-\infty}^{t} \varphi(x)dx = \lim_{\lambda \to +0} \left(\lambda \sum_{n=0}^{\infty} \varphi(t-\lambda n) \right).$$

The algebra B_0 contains the function $\ln p$, so that it is not longer necessary to exclude the case $\varphi_{-1} = 0$ in the integral of point 3^0, if we define

$$\int \frac{1}{p}\, dp = \ln p + C,$$

where C is the Euler constant. The definition of this integral for the remaining elements of B_0 can be done by means of a Hamel basis (cf. [3]). If we also consider translations $\Phi(p+\lambda)$ of the elements $\Phi(p) \in B_0$ with arbitrary complex numbers λ, and multiplications by $e^{-\lambda t}$ in A_0 (remark that there is no one-to-one correspondence between them), we get an operational calculus of almost the same generality as in the book of Amerbaev [1].

5^0. If we want to have an operator field, then we can either restrict B^* to the subfield of all functions which have for $p = 0$ at most a pole, or we can extend B^* to the corresponding quotient field, since B^* has no zero divisors. Of course, the same is possible with respect to B_0, and for all these fields there exist isomorphic fields connected with A^*.

The application of the foregoing statements for solving equations may be done in usual way, so we omit here this question.

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