DEMONSTRATIO MATHEMATICA
Vol. IX No 4 1976

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# ESTIMATION OF THE MODULUS AND ARGUMENT OF A QUASI-$\beta$-STARLIKE FUNCTION

1. Let $\mathcal{P}(\beta)$ denote the family of functions $P(z) = 1 + p_1 z + p_2 z^2 + \dots$ holomorphic in the disc $K = \{z : |z| < 1\}$ satisfying the condition

$$(1.1) \qquad \left|\frac{P(z)-1}{P(z)+1}\right| < \beta, \qquad 0 < \beta \leqslant 1, \quad z \in K.$$

By $\mathcal{P}$ we denote the class of functions $p(z) = 1+a_1z+a_2z^2+\dots$ holomorphic in $K$ and such that $\mathrm{re}\, p(z) > 0$ for $z \in K$. As shown in [3] the following lemmas hold.

Lemma 1. A function $P$ belongs to $\mathcal{P}(z)$ if and only if there exists a function $p \in \mathcal{P}$ satisfying the condition

$$(1.2) \qquad P(z) = \frac{(1+\beta)p(z)+1-\beta}{(1-\beta)p(z)+1+\beta}, \qquad z \in K.$$

Lemma 2. Let $z_0$ be any fixed point of $K$. We define the functional $H(P) = P(z_0)$, $P \in \mathcal{P}(\beta)$. Then the range of this functional is the closed disc with center $c$ and radius $\varrho$, where

$$(1.3) \qquad c = \frac{1+\beta^2 r^2}{1-\beta^2 r^2}, \quad \varrho = \frac{2\beta r}{1-\beta^2 r^2}, \quad r = |z_0|.$$

2. Let $S^*(\beta)$ denote the class of $\beta$-starlike functions i.e. functions $F$ holomorphic in $K$ such that

$$F(0) = 0, \quad F'(0) = 1 \quad \text{and} \quad \frac{zF'(z)}{F(z)} \in \mathcal{P}(\beta) \quad \text{for } z \in K.$$

Since $\mathcal{P}(1) \equiv \mathcal{P}$, we have $S^*(1) = S^*$, where $S^*$ is the well-known class of starlike functions.

D e f i n i t i o n. We say that $f$ is a quasi-$\beta$-starlike function if it satisfies the equation

$$F(f) = \frac{1}{M} F(z), \qquad z \in K, \tag{2.1}$$

where $F$ is some function in $S^*(\beta)$ and $M$ a fixed number in the interval $\langle 1, +\infty)$. The class of quasi-$\beta$-starlike functions will be denoted by $G^M(\beta)$.

For $\beta = 1$ we obtain the class of quasi-starlike functions [2].

Let $M = e^t$, $0 \leqslant t < \infty$, and let $f(z,t)$ be a quasi-$\beta$--starlike function defined by the equation

$$F(f) = e^{-t} F(z), \quad z \in K, \quad F \in S^*(\beta), \quad 0 \leqslant t < \infty. \tag{2.2}$$

It is easy to see that if $F \in S^*(\beta)$ is a fixed $\beta$-starlike function and $f(z,t)$ satisfies eqation (2.2), then we have

$$\lim_{t \to \infty} e^t f(z,t) = F(z), \tag{2.3}$$

$$\lim_{t \to \infty} e^t f'_z(z,t) = F'(z). \tag{2.4}$$

From (1.1) it follows that if $P \in \mathcal{P}(\beta)$, then $\frac{1}{P} \in \mathcal{P}(\beta)$. Making use of this remark, similarly as in [1], [4], we can prove the following theorems:

T h e o r e m 1. Each function $f \in G^M(\beta)$, $M = e^T$, can be represented in the form $f(z) = f(z,t)$, where $f(z,t)$ is the solution of the equation

$$\frac{\partial f(z,t)}{\partial t} = -f(z,t)P(f(z,t)), \qquad 0 \leqslant t \leqslant T, \tag{2.5}$$

satisfying the initial condition $f(z,0) = z$, where $P$ is a function in $\mathcal{P}(\beta)$.

T h e o r e m 2. If $P \in \mathcal{P}(\beta)$, then the function $f$ defined by the formula $f(z) = f(z,T)$, where $f(z,t)$ is the solution of equation (2.5) with the condition $f(z,0) = z$, belongs to family $G^M(\beta)$, $M = e^T$.

3. Now we shall give some application of Theorems 1 and 2 to the solving of extremal problems in the class $G^M(\beta)$.

T h e o r e m 3. If $f \in G^M(\beta)$, then for $|z| = r < 1$ we have the following exact estimation

$$\underline{x} \leqslant |f(z)| \leqslant \overline{x}, \tag{3.1}$$

where

$$\underline{x} = \frac{M(1+\beta r)}{2\beta^2 r}\left(1+\beta r - \sqrt{(1+\beta r)^2 - \frac{4\beta r}{M}}\right) - \frac{1}{\beta}. \tag{3.2}$$

$$\overline{x} = \frac{M(1-\beta r)}{2\beta^2 r}\left(1-\beta r - \sqrt{(1-\beta r)^2 + \frac{4\beta r}{M}}\right) + \frac{1}{\beta} \tag{3.3}$$

and $M = e^T$.

P r o o f. From equation (2.5) we obtain

$$d \log f(z,t) = -P(f(z,t))dt. \tag{3.4}$$

Equation (3.4) is equivalent to the system of equations

$$d \log |f(z,t)| = -\operatorname{re} P(f(z,t))dt, \tag{3.5}$$

$$d \arg f(z,t) = -\operatorname{im} P(f(z,t))dt. \tag{3.6}$$

From Lemma 2 we obtain

$$\frac{1-\beta r}{1+\beta r} \leqslant \operatorname{re} P(z) \leqslant \frac{1+\beta r}{1-\beta r}, \quad |z| = r < 1. \tag{3.7}$$

Hence by (3.5) we have

$$-\frac{1+\beta x}{1-\beta x}\,dt \leqslant d \log x \leqslant -\frac{1-\beta x}{1+\beta x}\,dt, \tag{3.8}$$

where $x = |f(z,t)|$, $z \in K$. Integrating (3.8) in the interval $[0,T]$, where $T = \log M$, and taking into account the initial condition we obtain

$$(3.9) \quad \frac{|f|}{(1+\beta|f|)^2} \geqslant \frac{1}{M}\frac{r}{(1+\beta r)^2}, \quad \frac{|f|}{(1-\beta|f|)^2} \leqslant \frac{1}{M}\frac{r}{(1-\beta r)^2},$$

where $f(z) = f(z,T) = f$, $|z| = r$. From (3.9) we get the thesis of Theorem 3. The estimation (3.1) is sharp. In fact, the equalities in (3.7) holds respectively for the functions

$$P_1(z) = \frac{1-\beta\varepsilon z}{1+\beta\varepsilon z}, \quad P_2(z) = \frac{1+\beta\varepsilon z}{1-\beta\varepsilon z}, \quad \text{where} \quad \varepsilon = e^{-i\varphi}.$$

Hence in view of (3.8) we infer that the modulus of a quasi--starlike function attains its maximum for the function satisfying the equation

$$\frac{f}{(1-\beta f)^2} = \frac{1}{M}\frac{z}{(1-\beta z)^2},$$

and the minimum of the modulus is attained for the function satisfying the equation

$$\frac{f}{(1+\beta f)^2} = \frac{1}{M}\frac{z}{(1+\beta z)^2}.$$

For $\beta = 1$ we obtain an estimation for the modulus of a quasi-starlike function ([1]).

T h e o r e m 4. If $f \in G^M(\beta)$, then we have

$$(3.10) \quad \left|\arg\frac{f(z)}{z}\right| \leqslant \log\frac{1-\beta|f|}{1+\beta|f|}\cdot\frac{1+\beta r}{1-\beta r}, \quad |z| = r,$$

and the equalities in (3.10) hold respectively for the functions defined by the equations

$$(3.11) \quad \frac{f}{(1-i\beta f)^2} = \frac{1}{M}\frac{z}{(1-i\beta z)^2}$$

$$(3.12) \quad \frac{f}{(1+i\beta f)^2} = \frac{1}{M}\frac{z}{(1+i\beta z)^2}.$$

P r o o f. From (3.5) and (3.6) it follows that

$$(3.13)\qquad d \arg f(z,t) = \frac{\operatorname{im} P(f(z,t))}{\operatorname{re} P(f(z,t))}\, d \log |f(z,t)| .$$

Putting $P(z) = u + iv$ for $z \in K$ and using Lemma 2 we obtain

$$(3.14)\qquad h(u,v) = \frac{\operatorname{im} P(z)}{\operatorname{re} P(z)} = \frac{v}{u}, \qquad (u,v) \in D,$$

where

$$(3.15)\qquad D = \left\{ (u,v) : (u-c)^2 + v^2 \leqslant \varrho^2 \right\},$$

and $c, \varrho$ are defined by (1.3).

From (3.14) we obtain $h'_u = -\frac{v}{u^2}$, $h'_v = \frac{1}{u}$. Since $h'_u \neq 0$ and $h'_v \neq 0$ for $(u,v) \in \operatorname{Int}(D)$, the function $h$ has no extremum inside the domain $D$.

Let $(u,v)$ be any boundary point of $D$ and let us assume

$$h_1(u) = h(u,v(u)) = \frac{\sqrt{\varrho^2-(u-c)^2}}{u} \quad \text{when} \quad v \geqslant 0,$$

$$h_2(u) = h(u,v(u)) = \frac{\sqrt{\varrho^2-(u-c)^2}}{u} \quad \text{when} \quad v < 0,$$

where $c-\varrho \leqslant u \leqslant c+\varrho$. It is easy to verify that the function $h_1$ attains the greatest value in the interval $\langle c-\varrho, c+\varrho \rangle$ at the point $u = 1/c$ and this value equals $\varrho$. The function $h_2$ attains at this point its least value equal to $-\varrho$. Since the function $h$ is continuous, we infer that the greatest value of $h$ in $D$ is equal to $\varrho$, and the smallest value to $\varrho$, where $\varrho$ is defined by (1.3). Hence from (3.14) we obtain

$$(3.16)\qquad \frac{-2\beta r}{1-\beta^2 r^2} \leqslant \frac{\operatorname{im} P(z)}{\operatorname{re} P(z)} \leqslant \frac{2\beta r}{1-\beta^2 r^2}, \qquad |z| = r .$$

The estimation (3.16) is sharp and the maximum is realized by the function

(3.17) $$P_1(z) = \frac{1+\beta iz}{1-\beta iz}$$

and the minimum by the function

(3.18) $$P_2(z) = \frac{1-\beta iz}{1+\beta iz}.$$

From (3.5), (3.13) and (3.16) we obtain

(3.19) $$\frac{2\beta x}{1-\beta^2x^2} d \log x \leqslant d \arg f(z,t) \leqslant$$

$$\leqslant \frac{-2\beta x}{1-\beta^2x^2} d \log x, \qquad x = \left|f(z,t)\right|.$$

Integrating (3.19) in the interval from $0$ to $T = \log M$ and taking into account the condition $f(z,0) = z$, we obtain

(3.20) $$\log \frac{1+\beta|f|}{1-\beta|f|} \cdot \frac{1-\beta r}{1+\beta r} \leqslant \arg \frac{f(z,T)}{z} \leqslant \log \frac{1-\beta|f|}{1+\beta|f|} \cdot \frac{1+\beta r}{1-\beta r}.$$

From (3.17) and (3.18) we infer that the function realizing the equalities (3.20) are of the form (3.11) and (3.12), respectively. This ends the proof of Theorem 4.

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Received January 15, 1976.