DEMONSTRATIO MATHEMATICA
Vol. IX No 4 1976

Urszula Sztaba

# SOME PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

1. We shall consider equations of the form

$$(E) \qquad y^{(n)}(t) + F(t,y(\tau(t)),y'(\tau(t)),\dots y^{(n-1)}(\tau(t))) = 0$$

and

$$(E_g) \qquad y^{(n)}(t) + g(t)f(t,y(t),y(\tau(t))) = 0,$$

where $g$ belongs to a family of functions $G$ having some property $W$. By a solution of equation (E) (resp. $(E_g)$) we shall understand every function of class $C^n$ satisfying (E) (resp. $(E_g)$) for sufficiently large $t$. Let $S_E$ denote the set of solutions of equation (E), and $S_{E_g}$ the set of solutions of $(E_g)$. We assume that $S_E \neq \emptyset$ and $S_{E_g} \neq \emptyset$. Let $Py$ be a propositional function defined on $S_E \cup S_{E_g}$ and let

$$g_y(t) \overset{df}{=} \frac{F(t,y(\tau(t)),y'(\tau(t)),\dots y^{(n-1)}(\tau(t)))}{f(t,y(t),y(\tau(t)))}$$

Staicos and Sficas [1] have formulated the following theorem.

Theorem 1.1. We have

$$\left[\forall_{g\in G}\ \forall_{y\in S_{E_g}} Py\right]\wedge\left[\exists_{\Sigma\subset S_E}\ \forall_{x\in\Sigma} \sim Px \Rightarrow g_x\in G\right]\Rightarrow\forall_{x\in\Sigma} Px.$$

The present work is based upon Theorem 1.1. In Section 2 we give sufficient conditions for the oscillation of all solutions of equation (E), in Section 3 sufficient conditions for the oscillation of all bounded solutions of equation (E). The following theorem ([3], [4], [5]) plays an essential role in our considerations.

T h e o r e m 1.2. If $q$ and $\tau$ are continuous for $t \geqslant 0$ such that $q(t) \geqslant 0$, $0 \leqslant \tau(t) \leqslant t$, $\lim_{t\to\infty} \tau(t) = \infty$ and

$$\int^{\infty} \left[\tau(t)\right]^{\alpha} q(t)dt = \infty \text{ where } \alpha = \begin{cases} \gamma(n-1) & \text{for } 0<\gamma<1 \\ n-1-\varepsilon,\ 0<\varepsilon<n-1 & \text{for } \gamma = 1 \end{cases}$$

or $\tau$ is non-increasing and $\int^{\infty} \left[\tau(t)\right]^{n-1} q(t)dt = \infty$ for $\gamma > 1$, then every solution of the equation

$$y^{(2n)}(t) + q(t)\left|y(\tau(t))\right|^{\gamma} \operatorname{sgn} y(t) = 0$$

is oscillating.

In Section 4 we investigate the properties of non-oscillating solutions of some second order differential equation. The following theorem is very helpful for this aim.

T h e o r e m 1.3. ([2] p. 37). If $a(t)$, $t_0 \leqslant t < \infty$, is a continuous monotonic function and $\lim_{t\to\infty} a(t) = c > 0$, then every solution of the equation $y''(t) + a(t)y(t) = 0$ is bounded.

2. Let $F(t,u_1,\dots,u_n)$ be defined and continuous in $D = \{(t,u_1,\dots,u_n): t \geqslant t_0,\ -\infty < u_i < \infty,\ i = 1\dots n\}$ such that $F(t,u_1,\dots,u_n)\, u_1 > 0$ for $u_1 \neq 0$. Let $\tau(t)$ be defined, non-increasing and continuous for $t \geqslant t_0$ such that $\tau(t) \leqslant t$ and $\lim_{t\to\infty} \tau(t) = \infty$. We assume that $n$ is even. For convenience we formulate further assumptions in the form of the following hypotheses.

H y p o t h e s i s $H_1$. There exists a number $\alpha < 1$ such that for every function $y(t)$ defined and continuous on $[t_y, \infty)$, $t_y \geqslant t_0$ with the properties

(i) $\lim\limits_{t \to \infty} y(t) \neq 0$

(ii) $y(t)y^{(n)}(t) \leqslant 0$ for sufficiently large $t$

we have

$$\int^{\infty} [\tau(t)]^{\alpha(n-1)} \frac{F(t, y(\tau(t)), \dots, y^{(n-1)}(\tau(t)))}{|y(\tau(t))|^{\alpha} \operatorname{sgn} y(t)} dt = \infty .$$

H y p o t h e s i s $H_2$. There exists $\varepsilon$, $0 < \varepsilon < n-1$, such that for every function $y(t)$ defined and continuous on $[t_y, \infty)$, $t_y \geqslant t_0$, with properties (i), (ii), we have

$$\int^{\infty} [\tau(t)]^{n-1-\varepsilon} \frac{F(t, y(\tau(t)), \dots, y^{(n-1)}(\tau(t)))}{y(\tau(t))} dt = \infty .$$

H y p o t h e s i s $H_3$. For every function $y(t)$ defined and continuous on $[t_y, \infty)$ with properties (i), (ii) we can select $\beta > 1$ such that we have

$$\int^{\infty} [\tau(t)]^{n-1} \frac{F(t, y(\tau(t)), \dots, y^{(n-1)}(\tau(t)))}{|y(\tau(t))|^{\beta} \operatorname{sgn} y(t)} dt = \infty .$$

Now we are going to prove the following theorem.

T h e o r e m 2.1. If any one of the hypotheses $H_1$, $H_2$, $H_3$ holds, then every solution of equation (E) oscillates.

P r o o f. We apply Theorem 1.1. Let $P_y$ mean: y oscillates. Suppose that $H_1$ holds. Let us denote

$$f(t, y(t), y(\tau(t))) = |y(\tau(t))|^{\alpha} \operatorname{sgn} y(t), \qquad 0 < \alpha < 1,$$

$$G = \Big\{ g(t) : g(t) \text{ defined and continuous on } [t_g, \infty),$$

$$g(t) \geqslant 0, \int^{\infty} [\tau(t)]^{\alpha(n-1)} g(t) dt = \infty \Big\} .$$

The validity of the first term of the predecessor of the implication results from Theorem 1.2. Assume that $\Sigma = S_E$. Observe that if $y(t) \in S_E$ and $y(t)$ does not oscillate, then $y(t)y^{(n)}(t) \leqslant 0$. Since $n$ is even, this implies $\lim_{t\to\infty} y(t) \neq 0$. From $H_1$ it follows that

$$g_y(t) = \frac{F(t,y(\tau(t)),\ldots,y^{(n-1)}(\tau(t)))}{|y(\tau(t))|^{\alpha} \operatorname{sgn} y(t)} \in G$$

which ends the proof of Theorem 2.1. in the case where $H_1$ holds.

Next assume that $H_2$ holds. We then take

$$f(t,y(t),y(\tau(t))) = y(\tau(t))$$

and

$$G = \Big\{ g : \text{the function } g(t) \text{ is defined and continuous on } [t_g,\infty),\ g(t) \geqslant 0,\ \exists_{0<\varepsilon<n-1} \int^{\infty} [\tau(t)]^{n-1-\varepsilon} g(t)dt = \infty \Big\} .$$

If $H_3$ holds, then we take

$$f(t,y(t),\ y(\tau(t))) = |y(\tau(t))|^{\beta} \operatorname{sgn} y(t), \quad \beta > 1$$

and

$$G = \Big\{ g : \text{the function } g(t) \text{ is defined and continuous on } [tg,\infty),\ g(t) \geqslant 0, \int^{\infty} [\tau(t)^{n-1} g(t)dt = \infty \Big\}.$$

In both cases we conclude the proof using Theorem 1.2, similarly as in the case $0 < \alpha < 1$.

R e m a r k 2.1. If we assume that $n$ is odd, then under the same assumptions we obtain that all solutions of equation (E) oscillate or tend to $0$ together with all derivatives up to the order $(n-1)$. If either $H_1$ or $H_2$ holds, we do not have to assume that the function $\tau$ is non-increasing.

R e m a r k 2.2. The class of equations to which Theorem 2.1. applies is wider than the class considered in [1], as the theorems of Staicos and Sficas cannot be applied in the case where for any $0 < a \leqslant 1$, we have $\lim_{t\to\infty} t^{-a}\tau(t)=0$. Theorem 2.1 implies Theorem 3.1 of [7] and Theorems 1 and 2 of [6].

3. H y p o t h e s i s $H_4$. For every function $y(t)$ defined and continuous on $[t_y,\infty)$, $t_y \geqslant t_0$, with properties (i), (ii) we have

$$\int^{\infty}[\tau(t)]^{n-1}F(t,y(\tau(t)),\dots,y^{(n-1)}(\tau(t)))dt=\infty .$$

T h e o r e m 3.1. If Hypothesis $H_4$ holds, then every bounded solution of equation (E) is oscillating.

P r o o f. Let

$$f(t,y(t),y(\tau(t)))=|y(\tau(t))|^{\beta}\operatorname{sgn} y(t), \quad \beta>1$$

$$G=\Big\{g(t) : g(t) \text{ defined and continuous on } [t_g,\infty), g(t)\geqslant 0, \int^{\infty}[\tau(t)]^{n-1}g(t)dt=\infty\Big\}.$$

We define $\Sigma$ to be the set of bounded solutions of the equation (E). Let $y(t)\in\Sigma$ and assume that $y(t)$ is not oscillating. From $H_4$ it follows that

$$[\tau(t)]^{n-1}\,\frac{F(t,y(\tau(t)),\dots,y^{(n-1)}(\tau(t)))}{|y(\tau(t))|^{\beta}\operatorname{sgn} y(t)}\,dt\geqslant$$

$$\geqslant\frac{1}{A^{\beta}}\int^{\infty}[\tau(t)]^{n-1}\,F(t,y(\tau(t)),\dots,y^{(n-1)}(\tau(t)))dt=\infty ,$$

$A = \text{const} \neq 0.$

Hence the function $\frac{F(t,y(\tau(t)),\dots,y^{(n-1)}(\tau(t)))}{|y(\tau(t))|^{\beta}\ \operatorname{sgn} y(t)}$ belongs to G, which was to be proved.

4. We shall consider the equations

$$(E') \qquad y''(t) + F(t,y(\tau_1(t)),y'(\tau_2(t))) = 0,$$

$$(E_g') \qquad y''(t) + g(t)y(t) = 0.$$

We make the following assumptions concerning the functions $F(t,u,v)$ and $\tau(t)$:

(1) $F(t,u,v)$ is defined and continuous in the set

$$\{(t,u,v) : t \geqslant t_0,\ -\infty < u,v < +\infty\}.$$

(2) $F(t,u,v)$ has the property that every solution of equation $(E')$ is extendable over the interval $[t_y, \infty)$.

(3) For every function $y(t)$ defined and continuous on $[t_y,\infty)$ which is not bounded and not oscillating we have

$$\lim_{t\to\infty} \frac{F(t,y(\tau_1(t)),y'(\tau_2(t)))}{y(t)} = c_y > 0$$

and $\frac{F(t,y(\tau_1(t)),y'(\tau_2(t)))}{y(t)}$ is monotonic for sufficiently large $t$.

(4) $\tau_i(t) \in C[t_0,\infty), \qquad i = 1,2.$

T h e o r e m 4.1. If assumptions (1) - (4) hold, then every non-oscillating solution of equation (E ) is bounded.

P r o o f. We apply Theorem 1.1 to equations $(E')$ and $(E_g')$. Let Py denote: $y$ is bounded on $[t_0,\infty)$. Let

$$g_y(t) \overset{df}{=} \frac{F(t,y(\tau_1(t)),y'(\tau_2(t)))}{y(t)},$$

$$G \stackrel{df}{=} \Big\{ g(t) : g(t) \text{ is defined, continuous and monotonic on } [t_g, \infty),\ \lim_{t\to\infty} g(t) = c_g > 0 \Big\}.$$

Let $\Sigma$ denote the set of all non-oscillating solutions of equation (E ). The validity of the predecessor of the implication in Theorem 1.1 results from Theorem 1.3. Let $y(t) \in \Sigma$ and assume that $\sim Py$. The function $g_y(t)$ is continuous for sufficiently large $t$, hence from (3) it follows that $g_y(t) \in G$.

D e f i n i t i o n 4.1. A function $h(t,u,v)$ is said to be almost monotonic in a set $D$ provided that

$$t_1 \leqslant t_2,\ |u_1| \leqslant |u_2|,\ |v_1| \geqslant |v_2|,\ u_1u_2 > 0,$$
$$v_1v_2 > 0,\ (t_i,u_i,v_i) \in D,$$
$$i = 1,2 \Longrightarrow h(t,u_1,v_1) \leqslant h(t,u_2,v_2)$$

or

$$t_1 \leqslant t_2,\ |u_1| \leqslant |u_2|,\ |v_1| \geqslant |v_2|,\ u_1u_2 > 0,$$
$$v_1v_2 > 0,\ (t_i,u_i,v_i) \in D,$$
$$i = 1,2, \Longrightarrow h(t,u_1,v_1) \geqslant h(t,u_2,v_2).$$

C o r o l l a r y 4.1. If

(5) $h(t,u,v)$ is defined, continuous, almost monotonic and non-negative in $D = \{ t \geqslant t_0,\ -\infty < u,v < +\infty \}$,

(6) $uh(t,u,v) \leqslant A_0 t + A_1|u| + A_2|v|$, $A_i = \text{const} \neq 0$, $i=0,1,2$,

(7) for every $a \in R$ we have

$$\lim_{\substack{t\to\infty \\ |u|\to\infty \\ v\to a}} h(t,u,v) = h_a > 0,$$

(8) $\tau(t) \in C[t_0,\infty)$, $\tau(t) \geqslant 0$, $\tau'(t) \geqslant 0$, $\lim_{t\to\infty} \tau(t) = \infty$

then every non-oscillating solution of the equation

$$y''(t) + y(t)h(t,y(\tau(t)), y'(\tau(t))) = 0$$

is bounded.

P r o o f. The accepted assumptions guarantee the existence of the solution on the positive semi-axis. Observe that if $y(t)$ is a non-bounded non-oscillating solution, then $y(t)y'(t) > 0$ and $y(t)y''(t) < 0$ for $t \geqslant t^* \geqslant t_0$. This implies that if $t_2 \geqslant t_1 \geqslant t^*$ then

$$|y(\tau(t_1)| \leqslant |y(\tau(t_2))|$$

and

$$|y'(\tau(t_1))| \geqslant |y'(\tau(t_2))|, \ h(t,y(\tau(t), y'(\tau(t)))$$

is monotonic for $t \geqslant t^*$. Since $\lim_{t\to\infty} |y(t)| = \infty$, from (7) it follows that $h(t,y(\tau(t)), y'(\tau(t))) = c > 0$.

## REFERENCES


[1] V.A. S t a i c o s, Y.G. S f i c a s: Oscillatory and asymptotic behavior of functional differential equations, J. Differential Equations, 12 (1972) 426-437.

[2] L. C e s a r i: Asymptotic behavior and stability problems in ordinary differential equations. New York 1971.

[3] В.М. Ш е в е л о, Н.В. В о р е х: Про каливальнисть розв'язкiв лiнiйних диффеpенцiальних рiвнянь вищих порядкiв iз запiзненням аргументу, Ukrain. Mat. Ž. 24 (1972) 511-518.

[4] В.М. Ш е в е л о: О некоторых свойствах решений дифференциальных уравнений с запаздыванием, Ukrain. Mat. Ž. 24 (1972) 807-813.

[5] В.М. Ш е в е л о: О влиянии запаздывания аргумента на колеблемость решений дифференциальных уравнений высшего порядка, Труды пятой международной конференции по нелинейным колебаниям. Kijev (1970) 553-557.

[6] R.S. D a h i y a, B. S i n g h: On oscillatory behavior of even order delay equations, J. Math. Anal. Appl. 42 (1973) 183-190.

[7] G. L a d a s: Oscillation and asymptotic behavior of solutions of differential equations with retarded argument, J. Differential Equations 10 (1971) 281-290.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE

Received July 30, 1975.