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THE EXISTENCE OF SOLUTIONS OF HEREDITARY SYSTEMS OF HYPERBOLIC TYPE

1. Introduction

In this paper we present the existence theorem for hereditary systems of hyperbolic type. A hereditary system is a system whose present state is in some way determined by its past history. Our result generalize, among others, the existence theorem given in J.K. Hale, M.A. Cruz [1].

2. Definitions and conventions

Before we pass to the basic problem of this paper, we introduce a notations to be used in the sequel. Denote by R the real line and by R^p a p-dimentional linear vector space with the norm $|\cdot|$ defined by $|x| = \sum_{k=1}^{p} |x_k|$ for $x = (x_1,\dots,x_p)$. Let D be a compact subset of R^2 and let us denote by $C_n(D)$ and $C_n^1(D)$ the spaces of all functions mapping D into R^n, respectively, continuous or continuous together with their first partial derivatives. We introduce in $C_n(D)$ and $C_n^1(D)$ norms respectively in the following way:

$$\|z\|_D = \sup_D |z(x,y)| \qquad \text{for} \quad z \in C_n(D)$$

and

$$\|w\|_D = \sup_D |w(x_1,x_2)| + \sup_D |w'_{x_1}(x_1,x_2)| + \sup_D |w'_{x_2}(x_1,x_2)|$$

for $w \in C_n^1(D)$.

We shall write $C(D)$ and $C^1(D)$ instead of $C_1(D)$ and $C_1^1(D)$ respectively.

Denote by E_1, E_2 open intervals of R and suppose $A_i = [-r_i, 0]$ $(i=1,2)$, where $r_i > 0$. We shall consider functions $\alpha_i : E_i \times A_i \longrightarrow R$ satisfying following conditions

(a) α_i, $\frac{\partial \alpha_i}{\partial x_i} \in C(E_i \times A_i)$ and $\left\| \frac{\partial \alpha_i}{\partial x_i} \right\|_{E_i \times A_i} < 1$ $(i = 1,2)$,

(b) $\alpha_i(x, A)$ are compact subsets of R for every fixed $x_i \in E_i$ $(i=1,2)$,

(c) $\alpha_i(x_i, \theta_i) \leqslant x_i$ for every $x_i \in E_i$, $\theta_i \in A_i$ $(i=1,2)$,

(d) $\alpha_i(x_i, \tilde{\theta}_i) \leqslant \alpha_i(x_i, \bar{\theta}_i)$ for every $x_i \in E$, $\tilde{\theta}_i$, $\bar{\theta}_i \in A_i$ such that $\tilde{\theta}_i \leqslant \bar{\theta}_i$ $(i=1,2)$,

(e) $\alpha_i(x_i, 0) = 0$ for every $x_i \in E_i$ $(i=1,2)$.

Let us denote by Ψ the set of all continuous functions $z : B_1 \times B_2 \longrightarrow R^n$, where $B_i = \alpha_i(E_i \times A_i)$ $(i=1,2)$. Let $E = E_1 \times E_2$ and $A = A_1 \times A_2$.

D e f i n i t i o n 1. The system $(A, \alpha_1, \alpha_2, H)$, where α_i $(i=1,2)$ satisfy (a) - (e) and $H : E \times \Psi \longrightarrow C_n(A)$ is defined by

$$(1) \qquad (H_{x_1 x_2} z)(\theta_1, \theta_2) = z(\alpha_1(x_1, \theta_1), \alpha_2(x_2, \theta_2))$$

for $z \in \Psi$, $x_i \in E_i$ and $\theta_i \in A_i$ $(i=1,2)$, will be called a hereditary structure.

Let $A = [-r_1, 0] \times [-r_2, 0]$ and let $(A, \alpha_1, \alpha_2, H)$ be a hereditary structure.

D e f i n i t i o n 2. The hereditary structure $(\bar{A}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{H})$ such that $\bar{A} = [-\bar{r}_1, 0]\ [-\bar{r}_2, 0]$, $\bar{\alpha}_i = \alpha_i | E \times [-\bar{r}_i, 0]$ and $(\bar{H}_{x_1 x_2} z)(\theta_1, \theta_2) = z(\bar{\alpha}_1(x_1, \theta_1), \bar{\alpha}_2(x_2, \theta_2))$, where $\bar{r}_i \leqslant r_i$, $x_i \in E_i$, $\theta_i \in A_i$ $(i=1,2)$ we shall call a substructure of $(A, \alpha_1, \alpha_2, H)$.

It will be convenient to write in the sequel x and y instead of x_1 and x_2 respectively.

Suppose (A,α_1,α_2,H) is a hereditary structure and let f and g be continunous functions mapping $E\times C_n(A)\times C_n(A)\times C_n(A)$ and $E\times C_n(A)$ respectively into R^n, where $E = E_1\times E_2$ and $A = A_1\times A_2$. Let $\mathcal{H}: C_n(A) \to C_n(E)$ be defined by

$$(\mathcal{H}\Phi)(x,y) = \Phi(0,0) - g(x,y,\Phi) \tag{2}$$

for $\Phi \in C_n(A)$ and $(x,y)\in E$.

In this paper we condider a hereditary system

$$\frac{\partial^2 \mathcal{H}(H_{xy}z)}{\partial x \partial y} = f\left(x,y,H_{xy}z, \frac{\partial H_{xy}z}{\partial x}, \frac{\partial H_{xy}z}{\partial y}\right). \tag{3}$$

Let us consider some special cases of (3). For $g = 0$, from (3) we obtain

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x,y,H_{xy}z, \frac{\partial H_{xy}z}{\partial x}, \frac{\partial H_{xy}z}{\partial y}\right). \tag{4}$$

Taking in (4) $\alpha_1(x,\Theta_1) = x_1 + \Theta_1$, $\alpha_2(y,\Theta_2) = y + \Theta_2$ for $\Theta_i \in A_i$ $(i=1,2)$ we get

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x,y,z(\cdot), \frac{\partial z(\cdot)}{\partial x}, \frac{\partial z(\cdot)}{\partial y}\right), \tag{5}$$

where $z(\cdot) = z(x+\Theta_1, y+\Theta_2)$.

If this simpler notation is again employed, we see that (3) includes the system of the form

$$\frac{\partial^2 \mathcal{H}(H_{xy}z)}{\partial x \partial y} = f\left(x,y,z(\cdot), \frac{\partial z(\cdot)}{\partial x}, \frac{\partial z(\cdot)}{\partial y}\right). \tag{6}$$

If $f = 0$, system (6) includes the difference equation

$$z(x,y) - g(x,y,z(x-\omega_1,y-\tau_1),\dots,z(x-\omega_k,y-\tau_k)) = h(x,y), \tag{7}$$

where $\omega_1,\dots,\omega_k \geqslant 0$, $\tau_1,\dots,\tau_k \geqslant 0$ and $h : E \to R^n$ is a given function.

In a similar way as in J.K. Hale, M.A. Cruz [1] we can see that (3) includes also Volterra integral equations of the form

$$z(x,y) = a(x,y) + \int_0^x\int_0^y b(x,y,u,v,z(u,v))dudv\,, \tag{8}$$

where a and b are given functions.

Let (A,α_1,α_2,H) be a hereditary structure and let $(\sigma,\tau)\in E_1\times E_2$, where E_i $(i=1,2)$ are open intervals of R. Let us denote by $E_{\sigma\tau}$ the set of the form

$$E_{\sigma\tau} = \bigcup_{\substack{x\geq\sigma \\ y\geq\tau}}\Big[(\overline{\mathrm{conv}}\,\alpha_1(x,A_1)\cap(-\infty,\sigma])\cup E_1\Big]\times$$

$$\times\Big[\overline{\mathrm{conv}}\,\alpha_2(y,A_2)\cap(-\infty,\tau]\Big]\cup\bigcup_{x\geq\sigma}\Big[\mathrm{conv}\,\alpha_1(x,A_1)\cap(-\infty,\sigma]\Big]\times E_2\,,$$

where $\overline{\mathrm{conv}}(G)$ denotes the closed convex hull of $G\subset R^n$. Suppose $\Phi\in C_n^1(E_{\sigma\tau})$.

D e f i n i t i o n 3. A function $z = z(\sigma,\tau,\Phi)$ is called the solution of (3) with initial function Φ at (σ,τ) if there exist numbers $\gamma_i > 0$ $(i=1,2)$ such that $z\in C_n^1(E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2})$, $z=\Phi$, $z'_x = \Phi'_x$, $z'_y = \Phi'_y$ on $E_{\sigma\tau}$, $\mathcal{H}(H_{xy}z)\in C_n^1(\Delta_{\gamma_1\gamma_2})$

$$\frac{\partial^2\mathcal{H}(H_{xy}z)}{\partial x\partial y}\in C_n(\Delta_{\gamma_1\gamma_2}) \quad\text{and}\quad \frac{\partial^2\mathcal{H}(H_{xy}z)}{\partial x\partial y} =$$

$$= f\left(x,y,H_{xy}z,\frac{\partial H_{xy}z}{\partial x},\frac{\partial H_{xy}z}{\partial y}\right)$$

for $(x,y)\in\Delta_{\gamma_1\gamma_2}$, where $\Delta_{\gamma_1\gamma_2} = [\sigma,\sigma+\gamma_1]\times[\tau,\tau+\gamma_2]$.

It is easy to see that (3) with initial function Φ at $(\sigma,\tau)\in E$ is equivalent to the following integral equation

$$(8) \qquad z(x,y) = \begin{cases} \phi(x,y) & \text{for } (x,y) \in E_{\sigma\tau} \\ \phi(x,\tau) + \phi(\sigma,y) - \phi(\sigma,\tau) + g(x,y,H_{xy}z) - \\ - g(x,\tau,H_{x\tau}z) - g(\sigma,y,H_{\sigma y}z) + g(\sigma,\tau,H_{\sigma\tau}z) + \\ + \int_\sigma^x\int_\tau^y f\left(u,v,H_{uv}z, \frac{\partial H_{uv}z}{\partial x}, \frac{\partial H_{uv}z}{\partial y}\right)dudv \\ & \text{for } (x,y) \in \Delta_{\gamma_1\gamma_2}. \end{cases}$$

3. Some fundamental lemmas

Let X be a Banach space and let Γ be a non-empty subset of X.

An operator $S : \Gamma \to X$ is said to be compact if it is continuous and maps bounded sets into precompact sets. We shall use the following slight generalisation of Schauder fixed point theorem due to Krasnoselskii [2] (see also J.K. Hale, M.A. Cruz [1]).

L e m m a 1. Suppose that Γ is a closed, bounded and convex subset of a Banach space X. If $T : \Gamma \to X$ is a contraction $S : \Gamma \to X$ is compact, $T(\Gamma) + S(\Gamma) = \{z = Tx + Sy : x,y \in \Gamma\} \subset \Gamma$ then $T + S$ has a fixed point in Γ.

Let (A,α_1,α_2,H) be a hereditary structure and let $L(C_A^1, R^n)$ denote the Banach space of all linear operators from $C_n^1(A)$ to R^n with the norm $\|\cdot\|$. Denote by $\mathcal{B}(E\times C_n^1(A))$ the set of all functions $g : E\times C_n^1(A) \to R^n$ continuous and bounded together with their partial derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ and such that the mapping $E\times C_n^1(A) \ni (x,y,\psi) \to (d_\psi g)(x,y,\psi) \in L(C_A^1,R^n)$, where $d_\psi g$ denotes the Fréchet derivative of $g(x,y,\psi)$ with respect to $\psi \in C_n^1(A)$, is continuous and bounded on $E\times C_n^1(A)$. Let $\lambda_g = \sup\left\{ \|(d_\psi g)(x,y,\psi)\| : (x,y,\psi) \in E\times C_n^1(A)\right\}$. For given positive number $\nu < \frac{1}{2}$ let us denote by Ω_ν the set of all functions $g \in \mathcal{B}(E\times C_n^1(A))$ such that $\lambda_g + 2\nu < 1$, $|g'_x(x,y,\phi) - g'_x(x,y,\psi)| \leqslant \nu \|\phi - \psi\|_A$, $|g'_y(x,y,\phi) +$

$- g'_y(x,y,\psi)| \leqslant \vartheta \ \|\phi-\psi\|_A$ and $\|(d_\psi g)(x,y,\phi)-(d_\psi g)(x,y,\psi)\| \leqslant \vartheta\|\phi-\psi\|_A$ for every $g\in\Omega_\vartheta$ and $\phi,\psi\in C_n^1(A)$. For given $\phi\in C_n(A)$, $(\sigma,\tau)\in E$ and $\gamma_1,\gamma_2>0$ by $\widetilde{\Phi}$ we shall mean the function defined on $E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2}$ by

$$\widetilde{\Phi}(x,y) = \begin{cases} \phi(x,y) & \text{for } (x,y)\in E_{\sigma\tau} \\ \phi(x,\tau)+\phi(\sigma,y)-\phi(\sigma,\tau) & \text{for } (x,y)\in\Delta_{\gamma_1\gamma_2}. \end{cases} \tag{9}$$

Let us denote by $\mathcal{F}$ the set of continuous and bounded functions $f : E\times C_n(A)\times C_n(A)\times C_n(A) \longrightarrow R$ and let $\|f\|_{\mathcal{F}} = \sup\{|f(x,y,\psi_1,\psi_2,\psi_3)| : (x,y)\in E,\ \psi_i\in C_n(A),\ i=1,2,3\}$.

L e m m a 2. Let (A,α_1,α_2,H) be a hereditary structure $\vartheta\in(0,\frac{1}{2})$, $f\in\mathcal{F}$, $g\in\Omega_\vartheta$ and $\phi\in C_n^1(A)$. Suppose that $|\phi|<1-\lambda_g-2\vartheta$. Then for every $(\sigma,\tau)\in E$ there are positive numbers γ_1,γ_2 and a hereditary substructure $(\bar{A},\bar{\alpha}_1,\bar{\alpha}_2,\bar{H})$ of (A,α_1,α_2,H) such that

(i)
$$|g(x,y,\bar{H}_{xy}\widetilde{\Phi})-g(\sigma,y,\bar{H}_{\sigma y}\widetilde{\Phi})| \leqslant K,$$
$$|g(x,\tau,\bar{H}_{x\tau}\widetilde{\Phi})-g(\sigma,y,\bar{H}_{\sigma y}\widetilde{\Phi})| \leqslant K,$$

(ii)
$$|g'_x(x,y,\bar{H}_{xy}\widetilde{\Phi})-g'_x(x,y,\bar{H}_{x\tau}\widetilde{\Phi})| \leqslant K,$$
$$|g'_y(x,y,\bar{H}_{xy}\widetilde{\Phi})-g'_y(\sigma,y,\bar{H}_{\sigma y}\widetilde{\Phi})| \leqslant K,$$

(iii)
$$\|(d_\psi g)(x,y,\bar{H}_{xy}\widetilde{\Phi})-(d_\psi g)(\sigma,y,\bar{H}_{\sigma y}\widetilde{\Phi})\| \leqslant K,$$
$$\|(d_\psi g)(x,y,\bar{H}_{xy}\widetilde{\Phi})-(d_\psi g)(x,\tau,\bar{H}_{x\tau}\widetilde{\Phi})\| \leqslant K,$$

(iv)
$$\left\|\frac{\partial\bar{H}_{xy}\widetilde{\Phi}}{\partial x}-\frac{\partial H_{x\tau}\widetilde{\Phi}}{\partial x}\right\|_{\bar{A}} \leqslant K, \quad \left\|\frac{\partial H_{xy}\widetilde{\Phi}}{\partial y}-\frac{\partial H_{x\tau}\widetilde{\Phi}}{\partial y}\right\|_{\bar{A}} \leqslant K$$

for every $(x,y)\in\Delta_{\gamma_1\gamma_2}$, where
$$K=\frac{1}{8}\left[\frac{1}{4}\,(1-\lambda_2-2\vartheta-|\phi|)^2-\|f\|_{\mathcal{F}}(\gamma_1+\gamma_2+\gamma_1\gamma_2)\right]$$
and $\widetilde{\Phi}$ is defined by (9).

P r o o f. Let $A = [-s,0]\times[-t,0]$, where $s,t \geqslant 0$. Since $(\sigma,\tau)\in E$ and $E = E_1\times E_2$ is open in R^2, we can choose positive numbers $\bar{\gamma}_1,\bar{\gamma}_2$ such that $\Delta_{\bar{\gamma}_1\bar{\gamma}_2} \subset E_1\times E_2$ and so that $(1-\lambda_\varepsilon - 2\nu - |\Phi|)^2 - 4\|f\|_{\mathcal{F}}(\bar{\gamma}_1+\bar{\gamma}_2+\bar{\gamma}_1\bar{\gamma}_2) > 0$. For $(\alpha_1(x,\theta_1),\ \alpha_2(y,\theta_2))\in\Delta_{\gamma_1\gamma_2}$ we have

$$\left(\frac{\partial H_{xy}\tilde{\Phi}}{\partial x}\right)(\theta_1,\theta_2) = \frac{\partial\Phi(\alpha_1(x,\theta_1),\alpha_2(y,\theta_2))}{\partial\alpha_1}\cdot\left(\frac{\partial\alpha_1}{\partial x}\right)(x,\theta_1),$$

$$\left(\frac{\partial H_{x\tau}\tilde{\Phi}}{\partial x}\right)(\theta_1,\theta_2) = \frac{\partial\Phi(\alpha_1(x,\theta_1),\alpha_2(\tau,\theta_2))}{\partial\alpha_1}\cdot\left(\frac{\partial\alpha_1}{\partial x}\right)(x,\theta_1),$$

$$\left(\frac{\partial H_{xy}\tilde{\Phi}}{\partial y}\right)(\theta_1,\theta_2) = \frac{\partial\Phi(\alpha_1(x,\theta_1),\alpha_2(y,\theta_2))}{\partial\alpha_2}\cdot\left(\frac{\partial\alpha_2}{\partial y}\right)(y,\theta_2),$$

and

$$\left(\frac{\partial H_{\sigma y}\tilde{\Phi}}{\partial y}\right)(\theta_1,\theta_2) = \frac{\partial\Phi(\alpha_1(\sigma,\theta),\alpha_2(y,\theta_2))}{\partial\alpha_2}\cdot\left(\frac{\partial\alpha_2}{\partial y}\right)(y,\theta_2).$$

By continuity of $\frac{\partial\Phi}{\partial\alpha_1}$, $\frac{\partial\Phi}{\partial\alpha_2}$, α_1 and α_2 it follows that there exist $\gamma_1\in(0,\bar{\gamma}_1)$, $\gamma_2\in(0,\bar{\gamma}_2)$, $\bar{t}>0$ and $\bar{s}>0$ such that

$$\left\|\frac{\partial H_{xy}\tilde{\Phi}}{\partial x} - \frac{\partial H_{x\tau}\tilde{\Phi}}{\partial x}\right\|_{\bar{A}} \leqslant K, \qquad \left\|\frac{\partial H_{xy}\tilde{\Phi}}{\partial y} - \frac{\partial H_{\sigma y}\tilde{\Phi}}{\partial y}\right\|_{\bar{A}} \leqslant K$$

for $(x,y)\in\Delta_{\gamma_1\gamma_2}$, where $\bar{A} = [-\bar{s},0]\ \ [-\bar{t},0]$.

Similarly, taking $\gamma_1,\gamma_2,\bar{s}$ and $\bar{t}$ sufficiently small, for $(\alpha_1(x,\theta_1),\alpha_2(y,\theta_2))\in\Delta_{\sigma\tau}$ and $(x,y)\in\Delta_{\gamma_1\gamma_2}$ we obtain

$$\left\|\frac{\partial H_{xy}\tilde{\Phi}}{\partial x} - \frac{\partial H_{x\tau}\tilde{\Phi}}{\partial x}\right\|_{\bar{A}} \leqslant K, \qquad \left\|\frac{\partial H_{xy}\tilde{\Phi}}{\partial y} - \frac{\partial H_{\sigma y}\tilde{\Phi}}{\partial y}\right\|_{\bar{A}} \leqslant K.$$

Let us observe that for a fixed $\psi \in C_n(E_{\sigma\tau} \cup \Delta_{\gamma_1\gamma_2})$ the set

$$\Lambda_\psi = \bigcup_{(x,y)\in\Delta_{\gamma_1\gamma_2}} \left\{\chi \in C_n(A) : \chi(\theta_1,\theta_2) = (H_{xy}\psi)(\theta_1,\theta_2)\right\}$$

is a compact subset of $C_n(A)$. Then, taking above $\gamma_1, \gamma_2, \bar{s}$ and $\bar{t}$ sufficiently small, we have $|g(x,y,H_{xy}\tilde{\Phi}) + - g(\sigma,y,H_{\sigma y}\tilde{\Phi})| \leqslant K$ for $(x,y) \in \Delta_{\gamma_1\gamma_2}$ and $(\theta_1,\theta_2) \in \bar{A}$.

In similar way we see that (i) - (iv) hold. This completes the proof.

4. Existence Theorem

Let $\mathcal{W}$ denote a non-empty subset of $C_n(E_{\sigma\tau} \cup \Delta_{\gamma_1\gamma_2})$, where $\gamma_1, \gamma_2 > 0$ and let

$$\Lambda_{\mathcal{W}} = \bigcup_{\substack{(x,y)\in\Delta_{\gamma_1\gamma_2} \\ \psi\in\mathcal{W}}} \left\{\chi \in C_n(A) : \chi = H_{xy}\psi\right\}$$

It is easy to see that $\Lambda_{\mathcal{W}}$ is a compact subset of $C_n(A)$ whenever $\mathcal{W}$ is compact in $C_n(E_{\sigma\tau} \cup \Delta_{\gamma_1\gamma_2})$.

For a given $g \in \Omega_\nu$ and $\delta > 0$ let

$$\omega_g(\delta) = \sup\left\{|g(x,y,\psi) - g(\bar{x},\bar{y},\bar{\psi})| : (x,y),\ (\bar{x},\bar{y}) \in \Delta_{\gamma_1\gamma_2},\ \psi,\bar{\psi} \in \Lambda_{\mathcal{W}} : |x-\bar{x}| \leqslant \delta,\ |y-\bar{y}| \leqslant \delta,\ \|\psi - \bar{\psi}\|_A \leqslant \delta\right\}.$$

It is clear that if $\mathcal{W}$ is a compact subset of $C_n(E_{\sigma\tau} \cup \Delta_{\gamma_1\gamma_2})$, then $\omega_g(\delta)$ is a continuous and non-decreasing function of $\delta \geqslant 0$ such that $\omega_g(0) = 0$.

In a similar way we can define the functions $\omega_{g'_x}(\delta)$, $\omega_{g'_y}(\delta)$, $\omega_{dg}(\delta)$, $\omega_{\alpha_1}(\delta)$ and $\omega_{\alpha_2}(\delta)$ corresponding to $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, $d_\psi g$, $\frac{\partial \alpha_1}{\partial x}$ and $\frac{\partial \alpha_2}{\partial y}$ respectively. Similarly, for $f \in \mathcal{F}$ and $\delta \geqslant 0$ we define the function $\omega_f(\delta)$ by

$$\omega_f(\delta) = \sup \Big\{ |f(x,y,\psi_1,\psi_2,\psi_3) - f(\bar{x},\bar{y},\bar{\psi}_1,\bar{\psi}_2,\bar{\psi}_3)| :$$

$$(x,y),(\bar{x},\bar{y}) \in \Delta_{\gamma_1\gamma_2}, \psi_i,\bar{\psi}_i \in \Lambda_W : |x-\bar{x}| \leqslant \delta ,$$

$$|y-\bar{y}| \leqslant \delta , \|\psi_i - \bar{\psi}_i\|_A \leqslant \delta ; \qquad i = 1,2,3 \Big\}$$

Let (A,α_1,α_2,H) be a hereditary structure and let $f \in \mathcal{F}$. Suppose that there exists a positive number L_f such that

$$(10) \qquad |f(x,y,\psi_1,\psi_2,\psi_3) - f(x,y,\psi_1,\bar{\psi}_2,\bar{\psi}_3)| \leqslant$$

$$\leqslant L_f(\|\psi_2 - \bar{\psi}_2\|_A + \|\psi_3 - \bar{\psi}_3\|_A)$$

for every $(x,y) \in E$ and $\psi_i,\bar{\psi}_i \in C_n(A)$ $(i=1,2,3)$.

For a given $f \in \mathcal{F}$ satisfying (10), $g \in \Omega_\nu$, $(\sigma,\tau) \in E$, $\Phi \in C_n(E_{\sigma\tau})$, $\gamma_1\gamma_2 > 0$ and a hereditary structure (A,α_1,α_2,H) let $P_r(\delta)$, $Q_r(\delta)$, $U_r(\delta)$ and $V_r(\delta)$ be defined by

$$(11) \qquad P_r(\delta) = \omega_{g'_x}(\delta) + \left[\nu r(1 + r + |\Phi|_{E_{\sigma\tau}} + \|f\|_{\mathcal{F}}\right]\delta +$$

$$+ \nu(1 + r + |\Phi|_{E_{\sigma\tau}})\omega_\Phi(\delta) + \lambda_g\omega_{g'_x}(\delta) + (r + |\Phi|_{E_{\sigma\tau}})\omega_{dg}(\delta) ,$$

$$(12) \qquad Q_r(\delta) = \omega_{g'_y}(\delta) + \left[\nu r(1 + r + |\Phi|_{E_{\sigma\tau}} + \|f\|_{\mathcal{F}}\right]\delta +$$

$$+ \nu(1 + r + |\Phi|_{E_{\sigma\tau}})\omega_\Phi(\delta) + \lambda_g\omega_{g'_y}(\delta) + (r + |\Phi|_{E_{\sigma\tau}})\omega_{dg}(\delta) ,$$

$$(13) \qquad U_r(\delta) = \lambda_g\omega_{g'_x}(\delta) + \nu r(1 + r + |\Phi|_{E_{\sigma\tau}})\delta +$$

$$+ \left[\nu(2 + r + |\Phi|_{E_{\sigma\tau}}) + \lambda_g|\Phi|_{E_{\sigma\tau}}\right]\omega_\Phi(\delta) + \frac{L_f\gamma_2}{1 - \lambda_g} Q_r(\delta) +$$

$$+ \left[\lambda_g(1 + |\Phi|_{E_{\sigma\tau}}) + L_f(r + |\Phi|_{E_{\sigma\tau}})\right]\omega_{\alpha_1}(\delta) + (r + 2|\Phi|_{E_{\sigma\tau}})\omega_{dg}(\delta) +$$

$$+ (\lambda_g + L_f\gamma_2)\omega_{\Phi'_x}(\delta) + L_f\gamma_2\omega_{\Phi'_y}(\delta) + \gamma_2\omega_f(\max\left[\delta, (r + |\Phi|_{E_{\sigma\tau}})\delta\right])$$

and

(14) $$V_r(\delta) = 2\,\omega_{g'_y}(\delta) + \vartheta r(1 + r + |\Phi|_{E_{\sigma\tau}})\delta +$$

$$+\left[\vartheta(2 + r + |\Phi|_{E_{\sigma\tau}}) + \lambda_g |\Phi|_{E_{\sigma\tau}}\right]\omega_\Phi(\delta) + \frac{L_f\gamma_1}{1-\lambda_g} P_r(\delta) +$$

$$+ \left[\lambda_g(1 + |\Phi|_{E_{\sigma\tau}}) + L_f(r + |\Phi|_{E_{\sigma\tau}})\right]\omega_{\alpha_2}(\delta) + (r + 2|\Phi|_{E_{\sigma\tau}})\,\omega_{dg}(\delta) +$$

$$+ (\lambda_g + L_f\gamma_1)\omega_{\Phi'_y}(\delta) + L_f\gamma_1\omega_{\Phi'_x}(\delta) + \gamma_1\,\omega_f(\max[\delta, (r + |\Phi|_{E_{\sigma\tau}})\delta]$$

for every $r > 0$ and $\delta > 0$.

Now, let Γ be a set of all functions $z \in C^1_n(E_{\sigma\tau} \cup \Delta_{\gamma_1\gamma_2})$ such that

(a) $z(x,y) = 0$ for $(x,y) \in E_{\sigma\tau}$, $|z|_{E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2}} \leqslant r$,

(b) $$\left|(\tfrac{\partial z}{\partial x})(x,y) - (\tfrac{\partial z}{\partial x})(x,\bar y)\right| \leqslant \frac{P_r(|y-\bar y|)}{1-\lambda_g},$$

(c) $$\left|(\tfrac{\partial z}{\partial y})(x,y) - (\tfrac{\partial z}{\partial y})(\bar x,y)\right| \leqslant \frac{Q_r(|x-\bar x|)}{1-\lambda_g},$$

(d) $$\left|(\tfrac{\partial z}{\partial x})(x,y) - (\tfrac{\partial z}{\partial x})(\bar x,y)\right| \leqslant \frac{U_r(|x-\bar x|)}{1-\lambda_g}\cdot \exp\left[\frac{L_f}{1-\lambda_g}(y-\tau)\right],$$

(e) $$\left|(\tfrac{\partial z}{\partial y})(x,y) - (\tfrac{\partial z}{\partial y})(x,\bar y)\right| \leqslant \frac{V_r(|y-\bar y|)}{1-\lambda_g}\cdot \exp\left[\frac{L_f}{1-\lambda_g}(x-\sigma)\right]$$

for $(\bar x,\bar y)$, $(x,y) \in \Delta_{\gamma_1\gamma_2}$, where $r = \frac{1}{2}\left[1 - \lambda_g - 2\vartheta + |\Phi|_{E_{\sigma\tau}}\right]$.

It is not difficult to verify that Γ is a compact convex subset of $C^1_n(E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2})$. Now, let us consider mappings $T,S : \Gamma \to C^1_n(E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2})$ defined by

$$(Tz)(x,y) = \begin{cases} 0 & \text{for } (x,y)\in E_{\sigma\tau} \\ g(x,y,H_{xy}(z+\tilde{\Phi}) - g(\sigma,y,H_{\sigma y}\tilde{\Phi}) + g(\sigma,\tau,H_{\sigma\tau}\tilde{\Phi}) - \\ -g(x,\tau,H_{x\tau}\tilde{\Phi}) & \text{for } (x,y)\in \Delta_{\gamma_1\gamma_2} \end{cases}$$

and

$$(Sz)(x,y) = \begin{cases} 0 & \text{for } (x,y)\in E_{\sigma\tau} \\ \int_\sigma^x\int_\tau^y f\left(u,v,H_{uv}(z+\tilde{\Phi}), \frac{\partial H_{uv}(z+\tilde{\Phi})}{\partial x}, \frac{\partial H_{uv}(z+\tilde{\Phi})}{\partial y}\right) dudv \\ & \text{for } (x,y)\in \Delta_{\gamma_1\gamma_2} \end{cases}$$

We shall prove the main result of our paper.

T h e o r e m 3. Let (A,α_1,α_2,H) be a hereditary structure, $\Phi\in C_n^1(A)$ $g\in\Omega_\vartheta$ and $f\in\mathcal{F}$. Suppose that $|\Phi|_A < 1-\lambda_g - 2\vartheta$ and f satisfies (10). Then there exist a hereditary substructure, say $(\bar{A},\bar{\alpha}_1,\bar{\alpha}_2,\bar{H})$ of (A,α_1,α_2,H), and positive numbers γ_1,γ_2 such that (3) corresponding to $(\bar{A},\bar{\alpha}_1,\bar{\alpha}_2,H)$, with the initial function Φ at $(\sigma,\tau)\in E$ has at least one solution $z^*\in C_n^1(E_{\sigma\tau}\cup\Delta_{\gamma_1\gamma_2})$.

P r o o f. By Lemma 2 there is a substructure $(\bar{A},\bar{\alpha}_1,\bar{\alpha}_2,H)$ and positive numbers γ_1,γ_2 such that (i) - (iii) of this Lemma hold. Let us observe that (3) with initial function Φ at (σ,τ) is equivalent to the equation $z=(T+S)(z)$ for $z\in\Gamma$. Therefore it suffices to prove that T and S satisfy the assumptions of Lemma 1.

For $z,w\in\Gamma$ and $(x,y)\in\Delta_{\gamma_1\gamma_2}$ we have

$$\left|(Tz)(x,y) - (Tw)(x,y)\right| \leqslant$$

$$\leqslant \left|g(x,y,H_{xy}(z+\tilde{\Phi})) - g(x,y,H_{xy}(w+\tilde{\Phi}))\right| \leqslant$$

$$\leqslant \lambda_g \left\|H_{xy}z - H_{xy}w\right\|_{\bar{A}} \leqslant \lambda_g \left|z-w\right|_{\Delta_{\gamma_1\gamma_2}}.$$

In a similar way we obtain

$$\left|(Tz)'_x(x,y) - (Tw)'_x(x,y)\right| \leq$$

$$\leq \left|g'_x(x,y,H_{xy}(z+\tilde{\Phi})) - g'_x(x,y,H_{xy}(w+\tilde{\Phi}))\right| +$$

$$+ \Big\| (d_\psi g)(x,y,H_{xy}(z+\tilde{\Phi})) \frac{\partial H_{xy}(z+\tilde{\Phi})}{\partial x} +$$

$$- (d_\psi g)(x,y,H_{xy}(w+\tilde{\Phi})) \frac{\partial H_{xy}(w+\tilde{\Phi})}{\partial x} \Big\| \leq$$

$$\leq \lambda_g \|z - w\|_{\Delta_{\gamma_1\gamma_2}} + \Big\|(d_\psi g)(x,y,H_{xy}(z+\tilde{\Phi})) \frac{\partial H_{xy}(z+\tilde{\Phi})}{\partial x} +$$

$$- (d_\psi g)(x,y,H_{xy}(z+\tilde{\Phi})) \frac{\partial H_{xy}(w+\tilde{\Phi})}{\partial x} \Big\| +$$

$$+ \Big\|(d_\psi g)(x,y,H_{xy}(z+\tilde{\Phi})) \frac{\partial H_{xy}(w+\tilde{\Phi})}{\partial x} +$$

$$- (d_\psi g)(x,y,H_{xy}(w+\tilde{\Phi})) \frac{\partial H_{xy}(w+\tilde{\Phi})}{\partial x} \Big\| \leq$$

$$\leq \lambda_g \|z - w\|_{\Delta_{\gamma_1\gamma_2}} + \vartheta \left\| \frac{\partial z}{\partial x} - \frac{\partial w}{\partial x} \right\|_{\Delta_{\gamma_1\gamma_2}} + \vartheta (r + |\Phi|_{\bar{A}} \|z - w\|_{\Delta_{\gamma_1\gamma_2}}$$

and

$$\left| (Tz)'_y(x,y) - (Tw)'_y(x,y)\right| \leq$$

$$\leq \vartheta \|z - w\|_{\Delta_{\gamma_1\gamma_2}} + \vartheta \left\| \frac{\partial z}{\partial y} - \frac{\partial w}{\partial y} \right\|_{\Delta_{\gamma_1\gamma_2}} + \vartheta (r + |\Phi|_A \|z - w\|_{\Delta_{\gamma_1\gamma_2}},$$

where $r = \frac{1}{2}\left[1 - \lambda_g - 2\vartheta - |\Phi|_A\right]$. Therefore

$$\|Tz - Tw\|_{\Delta_{\gamma_1\gamma_2}} \leqslant (1 - |\Phi|_A)\|z - w\|_{\Delta_{\gamma_1\gamma_2}}.$$

Since $|\Phi|_A \in (0,1)$ we infer that T is a contraction.

Now let us observe that if for every $z \in \Gamma$ the function

$$f\left(x,y,H_{xy}(z+\tilde{\Phi}), \frac{\partial H_{xy}(z+\tilde{\Phi})}{\partial x}, \frac{\partial H_{xy}(z+\tilde{\Phi})}{\partial y}\right)$$

is defined on the set $\Delta_{\gamma_1\gamma_2} \times \Lambda_z^3$ then for every $z \in \Gamma$ it is uniformly continuous on this set. Hence it is easy to see that S is continuous on Γ. Let $w = Sz$ for $z \in \Gamma$. For (x,y), $(\bar{x},\bar{y}) \in \Delta_{\gamma_1\gamma_2}$ we have

$$\left|w(x,y)\right| \leqslant \gamma_1\gamma_2\|f\|_{\mathfrak{F}}, \quad \left|(\tfrac{\partial w}{\partial x})(x,y)\right| \leqslant \gamma_2\|f\|_{\mathfrak{F}},$$

$$\left|(\tfrac{\partial w}{\partial y})(x,y)\right| \leqslant \gamma_1\|f\|_{\mathfrak{F}}, \quad \left|(\tfrac{\partial w}{\partial x})(x,y) - (\tfrac{\partial w}{\partial x})(x,\bar{y})\right| \leqslant \|f\|_{\mathfrak{F}}|y-\bar{y}|,$$

$$\left|(\tfrac{\partial w}{\partial y})(x,y) - (\tfrac{\partial w}{\partial y})(\bar{x},y)\right| \leqslant \|f\|_{\mathfrak{F}}|x-\bar{x}|,$$

$$\left|(\tfrac{\partial w}{\partial x})(\bar{x},y) - (\tfrac{\partial w}{\partial x})(x,y)\right| \leqslant \gamma_2\omega_f((1+|f|_{\mathfrak{F}}\gamma_2)|x-\bar{x}|) +$$

$$+ L_f\mu_1(y,|x-\bar{x}|) + L_f\omega_{\Phi'_x}(|x-\bar{x}|) + L_f(r+|\Phi|_A)\omega_{\alpha_1}(|x-\bar{x}|) +$$

$$+ L_f\left[\varkappa(|x-\bar{x}|) + \omega_{\Phi'_y}(|x-\bar{x}|)\right],$$

and

$$\left|(\tfrac{\partial w}{\partial y})(x,y)-(\tfrac{\partial w}{\partial y})(x,\bar{y})\right| \leqslant \gamma_1\omega_f((1+\gamma_1\|f\|_{\mathfrak{F}})|y-\bar{y}|)+L_f\mu_2(x,|y-\bar{y}|) +$$

$$+L_f\omega_{\Phi'_y}(|y-\bar{y}|)+L_f(r+|\Phi|_A)\omega_{\alpha_2}(|y-\bar{y}|)+L_f\left[\upsilon(|y-\bar{y}|)+\omega_{\Phi'_x}(|y-\bar{y}|)\right],$$

where

$$(15)\qquad \mu_1(y,\delta) = \frac{U_r(\delta)}{1-\lambda_g}\cdot \exp\left[\frac{L_f}{1-\lambda_g}\,(y-\tau)\right],\quad \varkappa(\delta) = \frac{Q_r(\delta)}{1-\lambda_g},$$

$$(16)\qquad \vartheta(\delta) = \frac{P_r(\delta)}{1-\lambda_g},\quad \mu_2(x,\delta) = \frac{V_r(\delta)}{1-\lambda_g}\cdot \exp\left[\frac{L_f}{1-\lambda_g}\,(x-\sigma)\right].$$

Therefore S is compact.

It remains to show that $T(\Gamma) + S(\Gamma) \subset \Gamma$. Let $z,w \in \Gamma$ and $u = Tw + Sz$. It is clear that $u(x,y) = 0$ for $(x,y) \in E_{\sigma\tau}$. For $(x,y) \in \Delta_{\gamma_1\gamma_2}$ we have

$$|u(x,y)| \leqslant |g(x,y,H_{xy}(w+\tilde{\Phi})) - g(x,y,H_{xy}\tilde{\Phi})| +$$

$$+ |g(x,y,H_{xy}\tilde{\Phi}) - g(\sigma,y,H_{\sigma y}\tilde{\Phi})| + |g(x,\tau,H_{\sigma\tau}\tilde{\Phi}) - g(x,\tau,H_{x\tau}\tilde{\Phi})| +$$

$$+ \|f\|_{\mathcal{F}}\gamma_1\gamma_2\|w\|_{\Delta_{\gamma_1\gamma_2}} + 2K + \|f\|_{\mathcal{F}}\gamma_1\gamma_2,$$

where K is defined in Lemma 2. In a similar way we get

$$\left|\left(\frac{\partial u}{\partial x}\right)(x,y)\right| \leqslant \vartheta\,\|w\|_{\Delta_{\gamma_1\gamma_2}} + \|w\|_{\Delta_{\gamma_1\gamma_2}}\left(\left\|\frac{\partial w}{\partial x}\right\|_{\Delta_{\gamma_1\gamma_2}} + \left\|\frac{\partial \Phi}{\partial x}\right\|_A\right) +$$

$$+ \lambda_g\left\|\frac{\partial w}{\partial x}\right\|_{\Delta_{\gamma_1\gamma_2}} + 3K + \|f\|_{\mathcal{F}}\gamma_2,$$

$$\left|\left(\frac{\partial u}{\partial y}\right)(x,y)\right| \leqslant \vartheta\,\|w\|_{\Delta_{\gamma_1\gamma_2}} + \|w\|_{\Delta_{\gamma_1\gamma_2}}\left(\left\|\frac{\partial w}{\partial y}\right\|_{\Delta_{\gamma_1\gamma_2}} + \left\|\frac{\partial \Phi}{\partial y}\right\|_A\right) +$$

$$+ \lambda_g\left\|\frac{\partial w}{\partial y}\right\|_{\Delta_{\gamma_1\gamma_2}} + 3K + \|f\|_{\mathcal{F}}\gamma_1.$$

Hence it follows that

$$|u|_{\Delta_{\gamma_1\gamma_2}} \leqslant |w|_{\Delta_{\gamma_1\gamma_2}} + \|w\|_{\Delta_{\gamma_1\gamma_2}}(|w|_{\Delta_{\gamma_1\gamma_2}} + |\Phi|_A) + 8K +$$

$$-2\nu|w|_{\Delta_{\gamma_1\gamma_2}} + |f|_{\mathcal{F}}(\gamma_1 + \gamma_2 + \gamma_1\gamma_2).$$

Putting $K = \frac{1}{8}\left[\frac{1}{4}(1-\lambda_g - 2\nu - |\Phi|_A)^2 - \|f\|_{\mathcal{F}}(\gamma_1+\gamma_2+\gamma_1\gamma_2)\right]$ and taking into account that $|w|_{\Delta_{\gamma_1\gamma_2}} \leqslant \frac{1}{2}(1-\lambda_g - 2\nu - |\Phi|_A)$ we obtain $|u|_{\Delta_{\gamma_1\gamma_2}} \leqslant r$, where $r = \frac{1}{2}(1 - \lambda_g - 2\nu - |\Phi|_A)$.

Let (x,y), $(\bar{x},\bar{y}) \in \Delta_{\gamma_1\gamma_2}$ and let $u = Tw + Sz$ for $w,z \in \Gamma$.

We have

$$\left|(\tfrac{\partial u}{\partial x})(\bar{x},y) - (\tfrac{\partial u}{\partial x})(\bar{x},y)\right| \leqslant$$

$$\left|(\tfrac{\partial g}{\partial x})(x,y,H_{xy}(w+\tilde{\Phi})) - (\tfrac{\partial g}{\partial x})(\bar{x},y,H_{xy}(w+\tilde{\Phi}))\right| +$$

$$+ \left|(\tfrac{\partial g}{\partial x})(\bar{x},y,H_{xy}(w+\tilde{\Phi})) - (\tfrac{\partial g}{\partial x})(x,y,H_{\bar{x}y}(w+\tilde{\Phi}))\right| +$$

$$+ \left\|d_\psi g)(x,y,H_{xy}(w+\tilde{\Phi}))\left[\frac{\partial H_{xy}(w+\tilde{\Phi})}{\partial x} - \frac{\partial H_{\bar{x}y}(w+\tilde{\Phi})}{\partial x}\right]\right\| +$$

$$+ \left\|\left[(d_\psi g)(x,y,H_{xy}(w+\tilde{\Phi})) - (d_\psi g)(\bar{x},y,H_{\bar{x}y}(w+\tilde{\Phi}))\right]\frac{\partial H_{\bar{x}y}(w+\tilde{\Phi})}{\partial x}\right\| +$$

$$+ \left\|\left[(d_\psi g)(\bar{x},y,H_{xy}(w+\tilde{\Phi})) - (d_\psi g)(\bar{x},y,H_{\bar{x}y}(w+\tilde{\Phi}))\right]\frac{\partial H_{\bar{x}y}(w+\tilde{\Phi})}{\partial x}\right\| +$$

$$+ \left\|(\tfrac{\partial g}{\partial x})(x,\tau,H_{x\tau}\tilde{\Phi}) - (\tfrac{\partial g}{\partial x})(\bar{x},\tau,H_{x\tau}\tilde{\Phi})\right\| +$$

$$+ \left|\left(\frac{\partial g}{\partial x}\right)(\bar{x},\tau,H_{x\tau}\tilde{\Phi}) - \left(\frac{\partial g}{\partial x}\right)(\bar{x},\tau,H_{\bar{x}\tau}\tilde{\Phi})\right| +$$

$$+ \left\|(d_\psi g)(x,\tau,H_{x\tau}\tilde{\Phi})\left[\frac{\partial H_{x\tau}\tilde{\Phi}}{\partial x} - \frac{\partial H_{\bar{x}\tau}\tilde{\Phi}}{\partial x}\right]\right\| +$$

$$+ \left\|\left[(d_\psi g)(\bar{x},\tau,H_{x\tau}\tilde{\Phi}) - (d_\psi g)(\bar{x},\tau,H_{\bar{x}\tau}\tilde{\Phi})\right]\frac{\partial H_{\bar{x}\tau}\tilde{\Phi}}{\partial x}\right\| +$$

$$+ \left\|\left[(d_\psi g)(\bar{x},\tau,H_{x\tau}\tilde{\Phi}) - (d_\psi g)(\bar{x},\tau,H_{\bar{x}\tau}\tilde{\Phi})\right]\frac{\partial H_{\bar{x}\tau}\tilde{\Phi}}{\partial x}\right\| +$$

$$+\int_\tau^y \left|f\left(x,t,H_{xt}(z+\tilde{\Phi}), \frac{\partial H_{xt}(z+\tilde{\Phi})}{\partial x}, \frac{\partial H_{xt}(z+\tilde{\Phi})}{\partial y}\right)+\right.$$

$$\left.- f\left(\bar{x},t,H_{\bar{x}t}(z+\tilde{\Phi}), \frac{\partial H_{\bar{x}t}(z+\tilde{\Phi})}{\partial x}, \frac{\partial H_{\bar{x}t}(z+\tilde{\Phi})}{\partial y}\right)\right| dt \leqslant$$

$$\leqslant U(|\bar{x}-x|) + \lambda_g\mu_1(y,|x-\bar{x}|) - L_f\int_\tau^y \mu_1(t,|x-\bar{x}|)dt ,$$

where $U(\delta)$ and $\mu_1(y,\delta)$ are defined by (13) and (15) respectively. Hence we obtain

$$\left|\left(\frac{\partial u}{\partial x}\right)(x,y) - \left(\frac{\partial u}{\partial x}\right)(\bar{x},y)\right| \leqslant \mu_1(y,|x-\bar{x}|)$$

for $(x,y)\in\Delta_{\gamma_1\gamma_2}$.

In a similar way for $(x,y)\in\Delta_{\gamma_1\gamma_2}$ we get

$$\left|\left(\frac{\partial u}{\partial x}\right)(x,y) - \left(\frac{\partial u}{\partial x}\right)(x,\bar{y})\right| \leqslant \vartheta(|y-\bar{y}|) ,$$

$$\left|\left(\frac{\partial u}{\partial y}\right)(x,y) - \left(\frac{\partial u}{\partial y}\right)(x,\bar{y})\right| \leqslant \mu_2(x,|y-\bar{y}|)$$

and

$$\left|\left(\frac{\partial u}{\partial y}\right)(x,y) - \left(\frac{\partial u}{\partial y}\right)(\bar{x},y)\right| \leqslant \varkappa(|x-\bar{x}|),$$

where ϑ, μ_2 and $\varkappa$ are defined by (15) and (16) respectively.

Hence we obtain $Tw + Sz \subset \Gamma$ for every $w,z \in \Gamma$. This completes the proof.

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