DEMONSTRATIO MATHEMATICA
Vol. IX No 4 1976

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SOLUTION OF THE FIRST FOURIER PROBLEM FOR THE GENERALIZED HEAT EQUATION AND ITS RELATION WITH LAPLACE EQUATION

Introduction

In this paper we construct a solution G for the equation

$$(1) \qquad L(x,D,D_t)u \equiv \Delta u - D_t u - V(x)u = 0,$$

where $x \in R_N$, $N > 2$, $t > 0$, and $V \in L^{\infty}(R_N)$ has a compact support. We investigate some properties of this solution and we solve the Fourier problem of the first kind for the equation

$$(1') \qquad L(x,D,D_t)u(x,t) = F(x,t).$$

1. The construction of the fundamental solution of the equation (1)

In a paper of Arseniev [1], a method analogous to one of E.E. Levi is presented for the construction of the fundamental solution $G(x,y,t)$ for the equation (1) under the assumption that $V(x)$ is a function of class $A^{-}(\alpha,R)$. Namely, the function G satisfies the integral equation

$$(2) \qquad G(x,y,t) = G_0(x,y,t) + \int_0^t d\tau \int_{R_N} G_0(x-\xi,t-\tau)V(\xi)G(\xi,y,\tau)d\xi,$$

where G_0 is the fundamental solution of the heat equation and has the form

$$(3) \qquad G_0(x-y,t) = (4\pi t)^{-N/2} \exp\left[-\frac{|x-y|^2}{4t}\right].$$

We seek a function G of the form

$$G(x,y,t) = G_0(x-y,t)\omega(x,y,t), \tag{4}$$

where $\omega \in L^{\infty}(R_N) \otimes L^{\infty}(R_N) \otimes L^{\infty}[0,T]$.

Then we obtain the following integral equation

$$\omega(x,y,t) = 1 + (A\omega)(x,y,t) \tag{5}$$

in which

$$(A\varphi)(x,y,t) = \tag{6}$$

$$= G_0^{-1}(x-y,t)\int_0^t d\tau \int_{R_N} G_0(x-\xi,t-\tau)V(\xi)G_0(\xi-y,\tau)\varphi(\xi,y,\tau)d\xi .$$

The solution ω of (5) can be represented as the limit of a uniformly convergent sequence $\{\omega_n\}$, where

$$\omega_n = \omega_0 + A\omega_{n-1} = \omega_0 + A\omega_0 + A^2\omega_0 + \dots + A^n\omega_0, \quad \omega_0 = 1. \tag{7}$$

In [1] it is shown that this solution of (5) is unique and satisfies the inequality

$$\sup_{\substack{x,y \in R_N \\ \tau \in [0,T]}} |\omega(x,y,\tau)| \leqslant C(T, \|V\|_{\infty}), \tag{8}$$

where the constant C is finite for $T \in [0,\infty)$, for the finite norm $\|V\|_{\infty}$.

In the present paper we shall find the fundamental solution for the equation (1) under the assumption that $V \in L^{\infty}(R_N) \in A^{-}(\alpha,R)$.

First we are going to prove the following lemmas.

L e m m a 1 (cf. [1], Lemma 1.1). If

1^0 $V \in L^{\infty}(R_N)$ and has a compact support,

2^0 $\sup_{x,y \in R_N} |\varphi(x,y,t)| \leqslant C_0 t^{\alpha}, \quad \alpha \geqslant 0,$

then we have

$$(9)\qquad \sup_{x,y\in R_N} \left|(A\varphi)(x,y,t)\right| \leqslant \frac{C_0}{\alpha+1} \|V\|_\infty t^{\alpha+1} .$$

P r o o f. From (3) and (6) in view of the hypothesis of the lemma it follows that

$$\left|(A\varphi)(x,y,t)\right| \leqslant C_0 \|V\|_\infty G_0^{-1}(x-y,t) \int_0^t \tau^\alpha \left[4\pi(t-\tau)4\pi\tau\right]^{-N/2} d\tau \cdot$$

$$\cdot \int_{R_N} \exp\left[-\frac{|x-\xi|^2}{4(t-\tau)} - \frac{|\xi-y|^2}{4\tau}\right] d\xi .$$

Substituting

$$s_j = \left(\frac{t}{4(t-\tau)\tau}\right)^{1/2} (\xi_j - y_j) + \left(\frac{\tau}{4(t-\tau)t}\right)^{1/2} (y_j - x_j), \quad j = 1,2,\dots,N$$

we obtain

$$\int_{R_N} \exp\left[-\frac{|x-\xi|^2}{4(t-\tau)} - \frac{|\xi-y|^2}{4\tau}\right] d\xi = \tau^{N/2} \left[\frac{4(t-\tau)}{t}\right]^{N/2} \cdot$$

$$\cdot \exp\left[-\frac{|x-y|^2}{4t}\right] \int_{R_N} \exp(-|s|^2) ds$$

and consequently we have

$$\left|(A\varphi)(x,y,t)\right| \leqslant C_0 \|V\|_\infty \int_0^t \tau^\alpha d\tau = \frac{C_0}{\alpha+1} \|V\|_\infty t^{\alpha+1}$$

for every $x \in R_N$ and $t \in [0,T]$. This ends the proof of the lemma.

L e m m a 2. If V belongs to $L^\infty(R_N)$ and has a compact support, then ω belongs to

$$L^\infty(R_N) \otimes L^\infty(R_N) \otimes L^\infty[0,T] \cap C^1(R_N \times R_N \times (0,T]), \qquad T > 0.$$

P r o o f. According to A.A. Arseniev (cf. [1], p.10) $\omega \in L^{\infty}(R_N) \otimes L^{\infty}(R_N) \otimes L^{\infty}[0,T]$. Observe that from (4) it follows that ω is in the class $C^1(R_N\times R_N\times(0,T])$ with respect to the variable y. It is known (see [4], p.82) that the function

$$G(x,y,t-\tau) = G_0(x-y,t-\tau)\omega(x,y,t-\tau)$$

is, as a function of the variables y,τ, the fundamental solution for the adjoint equation

$$\Delta_y v + D_\tau v - V(y)v = 0.$$

Hence this shows that G is of class $C^1(R_N\times R_N\times(0,T])$ with respect to y, and consequently ω is of the same class $C^1(R_N\times R_N\times(0,T])$ with respect to y.

2. The first Fourier problem for the equation (1')

We seek a function $u(x,t)$ satisfying the equation

(10) $$L(x,D,D_t)u \equiv \Delta u(x,t) - D_t u(x,t) - V(x)u(x,t) = F(x,t)$$

at every point $(x,t)\in \Omega_T \equiv \Omega\times(0,T)$, with the boundary condition

(11) $$\lim_{x\to\xi} u(x,t) = k(\xi,t), \quad (\xi,t)\in S\times(0,T) = \sigma_T,$$

(where S is the boundary of Ω) and with the initial condition

(12) $$\lim_{t\to 0^+} u(x,t) = f(x), \quad x\in\Omega.$$

We seek the solution $u(x,t)$ of class $C^2(\Omega_T)$ continuous in the closure $\bar{\Omega}_T \equiv \bar{\Omega}\times[0,T]$. We take the following assumptions:

1) The function $F(x,t)$ is defined and continuous in the domain $\Omega\times(0,T]$ and it satisfies Hölder's condition with respect to the space variable.

2) The function $k(\xi,t)$ is defined and continuous on the cylinder $\bar{\sigma}_T \equiv S \times [0,T]$, where S is a Lapunov surface.

3) The function $f(x)$ is defined and continuous in the domain $\bar{\Omega}$.

3. The solution of the problem

We seek the solution of the problem (10) - (12) in the form of a sum of three potentials

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)\,, \tag{13}$$

where

$$u_1(x,t) = -\int_0^t\int_\Omega G(x,t;y,\tau)F(y,\tau)\,dy\,d\tau\,, \tag{14}$$

$$u_2(x,t) = \int_\Omega G(x,t;y,0)f(y)\,dy\,, \tag{15}$$

$$u_3(x,t) = \int_0^t\int_S \frac{\partial G(x,t;\eta,\tau)}{\partial n_\eta}\,\varphi(\eta,\tau)\,d\eta\,d\tau\,, \tag{16}$$

and n_η denotes the normal at the point η to the surface S directed to the interior of the domain. The existence of the directional derivative $\frac{\partial G}{\partial n_\eta}$ under the sign of integral in formula (16) is secured by Lemma 2.

By formulas (4), (14), (15), (16) we can write the above integrals in the form

$$u_1(x,t) = -\int_0^t\int_\Omega G_0(x-y,t-\tau)\,\omega(x,y,t-\tau)F(y,\tau)\,dy\,d\tau\,, \tag{17}$$

$$u_2(x,t) = \int_\Omega G_0(x-y,t)\,\omega(x,y,t)f(y)\,dy\,, \tag{18}$$

$$(19)\qquad u_3(x,t) = \int_0^t\int_S \frac{\partial}{\partial n_\eta}\Big[G_0(x-\eta,t-\tau)\Big]\omega(x,\eta,t-\tau)\varphi(\eta,\tau)d\eta d\tau +$$

$$+\int_0^t\int_S G_0(x-\eta,t-\tau)\frac{\partial}{\partial n_\eta}\Big[\omega(x,\eta,t-\tau)\Big]\varphi(\eta,\tau)d\eta d\tau = J_1(x,t) + J_2(x,t).$$

We have the following lemma.

L e m m a 3. If the density $\varphi(\eta,\tau)$ is a continuous function in the domain $S\times(0,T]$, then the integral $u_3(x,t)$ defined by (16) has the following boundary value

$$(20)\qquad \lim_{x\to\xi\in S} u_3(x,t) = u_3(\xi,t) + \frac{1}{2}\varphi(\xi,t),$$

where the interior point x tends to ξ along to the normal passing through x, and $t\in(0,T]$.

P r o o f. The density $\omega\varphi$ in the integral J_1 satisfies the assumption of the theorem on jump (see [3], th. 3 p.15) and $\omega(\xi,\xi,0) = 1$. Hence the integral J_1 has the following property

$$(21)\qquad \lim_{x\to\xi\in S} J_1(x,t) =$$
$$=\int_0^t\int_S \frac{\partial}{\partial n_\eta}\Big[G_0(\xi-\eta,t-\tau)\Big]\omega(\xi,\eta,t-\tau)\varphi(\eta,\tau)d\eta d\tau + \frac{1}{2}\varphi(\xi,t).$$

Next we have

$$(22)\qquad \lim_{x-\xi\in S} J_2(x,t) =$$
$$=\int_0^t\int_S G_0(\xi-\eta,t-\tau)\frac{\partial}{\partial n_\eta}\Big[\omega(\xi,\eta,t-\tau)\Big]\varphi(\eta,\tau)d\eta d\tau.$$

From (21) and (22) we obtain the thesis (20).

Assuming that the function $u(x,t)$ satisfies the boundary condition (11) we obtain the integral equation

$$(23) \qquad \varphi(\xi,t) + 2\int_0^t\int_S \frac{\partial}{\partial n_\eta} G(\xi,t;\eta,\tau)\varphi(\eta,\tau)d\eta d\tau = g(\xi,t),$$

where we denote

$$(24) \qquad g(\xi,t) = 2k(\xi,t) + 2\int_0^t\int_\Omega G(\xi,t;y,\tau)F(y,\tau)dyd\tau +$$

$$+ 2\int_\Omega G(\xi,t;y,0)f(y)dy.$$

The function $g(\xi,t)$ is bounded and continuous on $\bar{\sigma}_T$ (by th. 7 and part 4 of the paper [3] and by lemma 2). Hence there exists a unique solution of the integral equation (23) defined by the formula

$$(25) \qquad \varphi(\xi,t) = -\int_0^t\int_S \mathcal{N}(\xi,t;\eta,\tau)g(\eta,\tau)d\eta d\tau + g(\xi,t),$$

where the solving kernel

$$(26) \qquad \mathcal{N}(\xi,t;\eta,\tau) = N(\xi,t;\eta,\tau) + \sum_{\nu=1}^{\infty}(-1)^\nu N_\nu(\xi,t;\eta,\tau)$$

is the sum of a series of iterated kernels defined by the recurrence formula

$$(27) \qquad N_{\nu+1}(\xi,t;\eta,\tau) = \int_\tau^t\int_S N(\xi,t;z,\zeta)N_\nu(z,\zeta;\eta,\tau)dzd\zeta,$$

$$(N_0 = N); \quad \nu = 0,1,\ldots,$$

where we have denoted

$$(28) \qquad N(\xi,t;\eta,\tau) = 2\frac{\partial G(\xi,t;\eta,\tau)}{\partial n_\eta}.$$

The series (26) is absolutely and uniformly convergent for $\xi \in S$ and $t \in (0,T)$. Hence taking into account the properties of heat potentials, it follows from (24) and (28) that the function $\varphi(\xi,t)$ is continuous with respect to ξ of S and to t (in agreement with part 6 of [3], formula 162). Substituting the function $\varphi(\xi,t)$ to the formula (13) we obtain the solution $u(x,t)$ of the problem (10) - (12).

4. A remark on the relation to Laplace's equation

Concluding this note we formulate the following problem.

Let u be a solution of the initial problem for (1) with the condition

$$u(x,0) = f(x)\,, \tag{29}$$

where f is a function fast decreasing at infinity (i.e. $f \in \mathcal{S}(R_N)$, cf. [2] p. 424).

If the fundamental solution G allows to integrate the function u with respect to t in the interval $(0,\infty)$, then we obtain a function of x. We state some property of this function.

In case $V = 0$ the equation (1) has the form

$$\Delta u(x,t) - D_t u(x,t) = 0. \tag{30}$$

For $N > 2$ the solution $u(x,t)$ of the problem (30), (29) can be integrated in the interval $(0,\infty)$ with respect to t. Then we obtain the following relation

$$\int_0^\infty u(x,t)dt = \int_0^\infty\int_\Omega G_0(x,t;y,0)f(y)dydt = \int_\Omega E(x,y)f(y)dy\,, \tag{31}$$

where $E(x,y)$ is the fundamental solution of Laplace's equation expressed by the formula

$$E(x,y) = \frac{1}{(N-2)\Theta_N|x-y|^{N-2}}, \qquad \text{for } N > 2\,, \tag{32}$$

where

$$\Theta_N = \frac{2(\sqrt{\pi})^N}{\Gamma(\frac{N}{2})}$$

is the area of the N-dimensional unit sphere.

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Received February 25, 1975.