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DENSITY CONCOMITANTS OF THE CURVATURE TENSOR $R_{\alpha\beta\gamma}{}^{\delta}$ IN THE PROJECTIVE-EUCLIDEAN SPACE

1. Introduction

The problem of determining algebraic concomitants for given geometric objects occupies a central place in the theory of geometric objects. Besides the analytical method [1], [5], which consists in reducing functional equations to systems of partial differential equations of first order, an algebraic method has been developed, in which one applies Jordan canonical form matrices and group theory. In the present paper we shall use an analytical method developed by S. Gołąb in [5]. Hence the results obtained will be valid for functions of class C^1.

Let $R_{\alpha\beta\gamma}{}^{\delta}$ denote the curvature tensor of the space L_n with a linear connection $\Gamma_{\alpha\beta}{}^{\gamma}$. Our aim is to determine all algebraic concomitants of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$, antysymmetric with respect to the indices α, β. These concomitants are densities with weight $(-r)$, (may be zero). Here we shall consider the case $n = 3$, but at the same time we restrict our problem to one of determining all density concomitants ot the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ satisfying some additional conditions. This paper is a continuations of [2] and [3].

2. Density with weight $(-r)$ as an algebraic concomitants of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$

First we shall give several basic symbols and definitions. If in a neighbourhood U of a point ξ_0 of the space L_n the transformation from the old coordinate system (λ) to a new one (λ') is described by a system of functions

(1) $$\xi^{\lambda'} = A^{\lambda'}(\xi^1,\dots,\xi^n), \qquad (\lambda' = 1',\dots,n'),$$

then $A^{\lambda'}_{\lambda}$ denotes as usual

(2) $$A^{\lambda'}_{\lambda} = \frac{\partial A^{\lambda'}(\xi^{\lambda})}{\partial \xi^{\lambda}}, \qquad (\lambda = 1,\dots,n).$$

For the transformation inverse to (1)

(3) $$\xi^{\lambda} = A^{\lambda}(\xi^{\lambda'}),$$

we assume the analogous notation

(4) $$A^{\lambda}_{\lambda'} = \frac{\partial A^{\lambda}(\xi^{\lambda'})}{\partial \xi^{\lambda'}}.$$

It is known that the elements (2) and (4) satisfy the condition

(5) $$A^{\lambda}_{\lambda'} = J^{-1} \text{ minor } A^{\lambda'}_{\lambda}, \text{ where } J = \det(A^{\lambda'}_{\lambda}) \neq 0,$$

and the minor $A^{\lambda'}_{\lambda}$ is understood in the algebraic sense. According to ([8], p.138), in the space L_n with an affine connection, the curvature tensor

(6) $$R_{\alpha\beta\gamma}{}^{\delta} = 2\partial_{[\alpha}\Gamma_{\beta]\gamma}{}^{\delta} + 2\Gamma^{\delta}_{[\alpha|\varrho|}\Gamma_{\beta]\gamma}{}^{\delta} \qquad (\alpha,\beta,\gamma,\delta,\varrho = 1,\dots,n)$$

can be interpreted as a differential concomitant of the first order for the object of parallel displacement $\Gamma_{\alpha\beta}{}^{\gamma}$. From (6) it follows that $R_{\alpha\beta\gamma}{}^{\delta}$ is an antisymmetric tensor with respect to α, β i.e.

(7) $$R_{\alpha\beta\gamma}{}^{\delta} = -R_{\beta\alpha\gamma}{}^{\delta}.$$

When we pass to the new coordinate system (λ') we find that the coordinates of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ satisfy the following transformation rule

(8) $$R_{\alpha'\beta'\gamma'}{}^{\delta'} = A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} A^{\gamma}_{\gamma'} A^{\delta'}_{\delta} R_{\alpha\beta\gamma}{}^{\delta}, \qquad \begin{pmatrix} \alpha,\beta,\gamma,\delta = 1,\dots,n \\ \alpha',\beta',\gamma',\delta' = 1',\dots,n' \end{pmatrix}.$$

In this section we shall not make use of the fact that the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ is a concomitant of the connection $\Gamma_{\alpha\beta}{}^{\gamma}$ of the form (6).

In the space L_n the number N of essential coordinates of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ satisfying the antisymmetry condition (7) is given by the formula $N = n^2\binom{n}{2}$ so that for $n = 3$, $N = 27$. Accordingly, in the coordinate system (λ) we assume the following brief notation:

$$(9)\quad \begin{cases} x_i=R_{121}{}^{i},\ x_{i+3}=R_{122}{}^{i},\ x_{i+6}=R_{123}{}^{i},\ y_i=R_{131}{}^{i},\ y_{i+3}=R_{132}{}^{i} \\ y_{i+6}=R_{133}{}^{i},\ z_i=R_{231}{}^{i},\ z_{i+3}=R_{232}{}^{i},\ z_{i+6}=R_{233}{}^{i},\ (i=1,2,3). \end{cases}$$

The sought algebraic concomitants $H(x_i,y_i,z_i)$ of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$, being a density of weight $(-r)$, satisfies the following functional equation when we pass the old coordinate system (λ) to the new one (λ'):

$$(10)\quad H(x_{i'},y_{i'},z_{i'}) = \varepsilon|J|^{r}\,H(x_i,y_i,z_i), \qquad (i = 1,\dots,9)$$

where

$$(11)\qquad \varepsilon = \begin{cases} 1, & \text{for W-density} \\ \operatorname{sgn} J, & \text{for G-density}, \end{cases}$$

The dependence of each of the variables $x_{i'}, y_{i'}, z_{i'}$ on the variables x_i,y_i,z_i is given by a formula of the type (8).

By an analytical method [5] the functional equation (13) can be replaced by a system of 9 partial differential equations with one unknown function H depending on 27 unknows x_i,y_i,z_i, $(i = 1,2,\dots,9)$ [2], [3].

The obtained system of equations is complete [3]. However, it is not integrable in any direction, neither according to the definition of A. Hoborski [6], nor according to the definition of K. Żorawski [10]. We shall solve the system of equations (18) - (26) in a special case, namely in the case where L_3 is the projective-euclidean space A_3.

3. The case of projective-euclidean space A_n

When the affine connection $\Gamma_{\alpha\beta}^{\gamma}$ is symmetric, i.e. when $\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}$, we call L_n the space with affine symmetric connection and denote it by A_n.

The space A_n is called projective-euclidean whenever there exists a vector field P_λ such that the space A_n with the connection object $\tilde{\Gamma}_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} + 2A_{(\alpha}^{\gamma} P_{\beta)}$ has the curvature tensor vanishing identically (see [4], or [8] p.288).

Let us introduce two tensors in the space A_n:

$$(12)\qquad P_{\alpha\beta} = \frac{1}{1-n^2}\left[(n+1)R_{\alpha\beta} + V_{\alpha\beta}\right] = \frac{1}{1-n^2}\,(n\,R_{\alpha\beta} + R_{\beta\alpha})$$

$$(13)\qquad W_{\alpha\beta\gamma}{}^{\delta} = R_{\alpha\beta\gamma}{}^{\delta} - 2\,P_{[\alpha\beta]}\,A_{\gamma}^{\delta} + 2\,A_{[\alpha}^{\delta} P_{\beta]\gamma},$$

where

$$(14)\qquad R_{\alpha\beta} = R_{\varrho\alpha\beta}{}^{\varrho}, \qquad V_{\alpha\beta} = R_{\alpha\beta\varrho}{}^{\varrho}.$$

By means of ([8], p.289) we have the following theorem.

T h e o r e m 1. (H.Weyl). For $n > 2$ the space A_n is projective euclidean if and only if the Weyl tensor $W_{\alpha\beta\gamma}{}^{\delta}$ vanishes

$$(15)\qquad W_{\alpha\beta\gamma}{}^{\delta} \equiv 0.$$

For $n > 2$ the condition (15) gives an algebraic relation between the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ and its algebraic concomitants $R_{\alpha\beta}$ and $V_{\alpha\beta}$.

In the space A_n the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ satisfies the Ricci identity of the form

$$(16)\qquad R_{[\alpha\beta\gamma]}{}^{\delta} = 0.$$

Instead of investigating the order of the system of equations (15) (with the unknowns $R_{\alpha\beta\gamma}{}^{\delta}$) we can examine the existence of a unique solution of the algebraic system of equations

(17) $$2P_{[\alpha\beta]}A^{\delta}_{\gamma} - 2A^{\delta}_{[\alpha}P_{\beta]\gamma} = R_{\alpha\beta\gamma}{}^{\delta}, \qquad (\alpha < \beta)$$

with respect to the unknowns $P_{\alpha\beta}$.

By the order of the system of equations (17) we understand the order of the projective-euclidean space A_n.

In virtue of (7) and (16) the system of equations (17) contains $L = \frac{n^2(n^2-1)}{3}$ independent equations and n^2 unknowns $P_{\alpha\beta}$. On the basis of ([8], p.290) and (17) we have the following corollary.

C o r o l l a r y 1. In the projective-euclidean space A_n the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ has, in the general case, at most n^2 independent coordinates. In the case of a connection preserving volume the number of independent coordinates is equal to $\binom{n+1}{2}$.

For $n = 3$ the indices $\alpha, \beta, \gamma, \delta$ take on values 1,2,3 (with limitation $\alpha < \beta$), from which by means of the corollary above and formula (17) we obtain 7 systems of equations determining $P_{\alpha\beta}$:

(18) $$R_{121}{}^{3} = R_{122}{}^{3} = R_{131}{}^{2} = R_{133}{}^{2} = R_{232}{}^{1} = R_{233}{}^{1} = 0.$$

(19) $$P_{11} = R_{121}{}^{2}, \qquad P_{11} = R_{131}{}^{3}.$$

(20) $$\begin{cases} P_{12} = R_{132}{}^{3}, \quad P_{21} = R_{231}{}^{3}, \quad P_{12} - 2P_{21} = R_{121}{}^{1}, \\ 2P_{12} - P_{21} = R_{122}{}^{2}, \quad P_{12} - P_{21} = R_{123}{}^{3}. \end{cases}$$

(21) $$\begin{cases} P_{13} = R_{123}{}^{2}, \quad -P_{31} = R_{231}{}^{2}, \quad P_{13} - 2P_{31} = R_{131}{}^{1} \\ 2P_{13} - P_{31} = R_{133}{}^{3}, \quad P_{13} - P_{31} = R_{132}{}^{2}. \end{cases}$$

(22) $$-P_{22} = R_{122}{}^{1}, \qquad P_{22} = R_{232}{}^{3}.$$

$$(23)\quad \begin{cases} -P_{23} = R_{123}{}^{1}, \quad -P_{32} = R_{132}{}^{1}, \quad P_{23} - 2P_{32} = R_{232}{}^{2} \\ 2P_{23} - P_{32} = R_{233}{}^{3}, \quad P_{23} - P_{32} = R_{231}{}^{1}. \end{cases}$$

$$(24)\qquad -P_{33} = R_{133}{}^{1}, \quad -P_{33} = R_{233}{}^{2}.$$

Solving the systems of equations (18) - (24) with respect to the indeterminates $P_{\alpha\beta}$ we obtain

$$(25)\quad \begin{cases} P_{11}=R_{121}{}^{2}, \; P_{12}=R_{132}{}^{3}, \; P_{13}=R_{123}{}^{2}, \; P_{21}=R_{231}{}^{3}, \\ P_{22}=R_{232}{}^{3}, \; P_{23}=R_{213}{}^{1}, \; P_{31}=R_{321}{}^{2}, \; P_{32}=R_{312}{}^{1}, \; P_{33}=R_{323}{}^{2}. \end{cases}$$

Moreover, besides conditions (18) the following three systems of conditions should hold

$$(26)\qquad R_{121}{}^{2} = R_{131}{}^{3}, \quad R_{122}{}^{1} = -R_{232}{}^{3}, \quad R_{133}{}^{1} = R_{233}{}^{2}.$$

$$(27)\quad \begin{cases} R_{123}{}^{3} = R_{132}{}^{3} - R_{231}{}^{3}, \quad R_{132}{}^{2} = R_{123}{}^{2} + R_{231}{}^{2}, \\ R_{231}{}^{1} = R_{132}{}^{1} - R_{123}{}^{1} \end{cases}$$

$$(28)\quad \begin{cases} R_{121}{}^{1} = -2R_{231}{}^{3} + R_{132}{}^{3}, \quad R_{122}{}^{2} = 2R_{132}{}^{3} - R_{231}{}^{3}, \\ R_{131}{}^{1} = 2R_{231}{}^{2} + R_{123}{}^{2}, \quad R_{133}{}^{3} = 2R_{123}{}^{2} + R_{231}{}^{2}, \\ R_{233}{}^{3} = -2R_{123}{}^{1} + R_{132}{}^{1}, \quad R_{232}{}^{2} = 2R_{132}{}^{1} - R_{123}{}^{1}. \end{cases}$$

By Ricci's identity (16) the fourth conditions of the systems of conditions (20), (21) and (23) are identities.

From the tensor character of formula (17) we infer that the systems of conditions (18) and (25) - (28) are invariant, i.e. independent of the coordinate system (λ).

In this way in the space A_3 among 27 coordinates of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ we have distinguished 9 essential coordinates. By formulas (18) and (26) - (28) the remaining 18 coordinates of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ depend on 9 essential ones.

4. The determination of the density (scalar) concomitant of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ in the projective-euclidean space A_3

Taking into account the invariant conditions (18), (25) - (28) for the coordinates of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ we introduce the following notation for the coordinates of the tensor $P_{\alpha\beta}$

(29)
$$\begin{cases} u_1 = P_{11},\ u_2 = P_{12},\ u_3 = P_{13},\ u_4 = P_{21},\ u_5 = P_{22}, \\ u_6 = P_{23},\ u_7 = P_{31},\ u_8 = P_{32},\ u_9 = P_{33}. \end{cases}$$

Hence in view of (29), (18), (25) - (28) all coordinates of the tensor $R_{\alpha\beta\gamma}{}^{\delta}$ (for $n = 3$) can be expressed by means of the parameters $u_1,\dots,u_9$, namely

(30)
$$\begin{cases} R_{121}{}^{2} = u_1,\ R_{132}{}^{3} = u_2,\ R_{123}{}^{2} = u_3,\ R_{231}{}^{3} = u_4,\ R_{232}{}^{3} = u_5 \\ R_{213}{}^{1} = u_6,\ R_{231}{}^{2} = u_7,\ R_{312}{}^{1} = u_8,\ R_{323}{}^{2} = u_9. \end{cases}$$

(31)
$$R_{121}{}^{3} = R_{122}{}^{3} = R_{131}{}^{2} = R_{133}{}^{2} = R_{232}{}^{1} = R_{233}{}^{1} = 0.$$

(32)
$$R_{131}{}^{3} = u_1,\quad R_{122}{}^{1} = -u_5,\quad R_{133}{}^{1} = -u_9.$$

(33)
$$R_{123}{}^{3} = u_2 - u_4,\ R_{132}{}^{2} = u_3 - u_7,\ R_{231}{}^{1} = u_6 - u_8.$$

(34)
$$\begin{cases} R_{121}{}^{1} = u_2 - 2u_4,\ R_{122}{}^{2} = 2u_2 - u_4,\ R_{131}{}^{1} = u_3 - 2u_7 \\ R_{133}{}^{3} = 2u_3 - u_7,\ R_{233}{}^{3} = 2u_6 - u_8,\ R_{232}{}^{2} = u_6 - 2u_8. \end{cases}$$

By (9), (10), (30) - (34) the sought concomitant $H(x_k, y_k, z_k)$ (k = 1,...,9) depends only upon the essential variables u_i (i=1,...,9).

Hence we assume the following notation

(35) $$G(u_i) = H(x_k, y_k, z_k), \qquad (i,k = 1,\dots,9).$$

As usual we denote

(36) $$G_i = \frac{\partial G}{\partial u_i} \qquad (i = 1,\dots,9).$$

By virtue of (9) as well as (30) - (36) the system of partial differential equations corresponding to the functional equation (10) (see [2], [3]) can be reduced to the following system of nine partial differential equations of first order (for the unknown function G depending upon nine variables $u_1,\dots,u_9$)

(37) $$\begin{cases} 2u_1G_1 + u_2G_2 + u_3G_3 + u_4G_4 + u_7G_7 = -rG \\ u_1G_2 + u_1G_4 + (u_2+u_4)G_5 + u_3G_6 + u_7G_8 = 0 \\ u_1G_3 + u_4G_6 + u_1G_7 + u_2G_8 + (u_3+u_7)G_9 = 0 \\ (u_2+u_4)G_1 + u_5G_2 + u_6G_3 + u_5G_4 + u_8G_7 = 0 \\ 2u_1G_1 + u_3G_3 - 2u_5G_5 - u_6G_6 + u_7G_7 - u_8G_8 = 0 \\ u_2G_3 + u_5G_6 + u_4G_7 + u_5G_8 + (u_6+u_8)G_9 = 0 \\ (u_3+u_7)G_1 + u_8G_2 + u_9G_3 + u_6G_4 + u_9G_7 = 0 \\ u_3G_2 + u_7G_4 + (u_6+u_8)G_5 + u_9G_6 + u_9G_8 = 0 \\ 2u_1G_1 + u_2G_2 + u_4G_4 - u_6G_6 - u_8G_8 - 2u_9G_9 = 0. \end{cases}$$

The system of equations (37) is not integrable in any direction, neither in the sense of A. Hoborski [6], nor in the sense of K. Żorawski [10]. In [6] and [10] methods are given for reducting an arbitrary complete linearly-independent system of first order partial differential equations to an equivalent system which is integrable in some direction. However, this method of reduction is very complicated. By a trial method we have ascertained that the system (37) is integrable in the direction of equations (37.8), (37.2), (37.7), (37.5), (37.9), (37.3), (37.4), (37.1), (37.6) ((37.k) denotes the k-th equation of the system (37)).

To the partial differential equation (37.8) there corresponds the following system of ordinary differential equations:

$$(38)\quad \begin{cases} \dfrac{du_1}{0} = \dfrac{du_2}{u_3} = \dfrac{du_3}{0} = \dfrac{du_4}{u_7} = \dfrac{du_5}{u_6+u_8} = \\ \dfrac{du_6}{u_9} = \dfrac{du_7}{0} = \dfrac{du_8}{u_9} = \dfrac{du_9}{0}. \end{cases}$$

This system has the following first integrals:

$$(39)\quad \begin{cases} C_1 = u_1, \quad C_2 = u_3, \quad C_3 = u_7, \quad C_4 = u_9, \quad C_5 = u_6 - u_8 \\ C_6 = u_6u_8 - u_5u_9, \; C_7 = u_2u_9 - u_3u_8, \; C_8 = u_4u_9 - u_6u_7. \end{cases}$$

Hence the general solution of (37.8) has the form

$$(40)\quad G = \varphi(u_1, u_3, u_7, u_9, u_6-u_8, u_6, u_8-u_5u_9, u_2u_9-u_3u_8, u_4u_9-u_6u_7) =$$

$$= \varphi(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8,$$

where φ is an arbitrary function of class C^1.

Substituting the solution (40) into the equation (37.2) we obtain the following equation $(\varphi_k = \partial\varphi/\partial v_k)$

$$(41)\quad (v_2-v_3)\varphi_5 - (v_7+v_8)\varphi_6 + (v_1v_4-v_2v_3)\varphi_7 + (v_1v_4-v_2v_3)\varphi_8 = 0.$$

To (41) there corresponds the following system of ordinary differential equations

$$(42)\quad \begin{cases} \dfrac{dv_1}{0} = \dfrac{dv_2}{0} = \dfrac{dv_3}{0} = \dfrac{dv_4}{0} = \\ = \dfrac{dv_5}{v_2 - v_3} = \dfrac{dv_6}{-v_7 - v_8} = \dfrac{dv_7}{v_1v_4 - v_2v_3} = \dfrac{dv_8}{v_1v_4 - v_2v_3} \end{cases}$$

whose first inregrals have the form

$$(43)\quad \begin{cases} D_1 = v_1, \quad D_2 = v_2, \quad D_3 = v_3, \quad D_4 = v_4, \quad D_5 = v_7 - v_8, \\ D_6 = v_2v_3v_5 - v_1v_4v_5 - v_3v_7 + v_2v_7, \; D_7 = v_2v_3v_6 - v_1v_4v_6 - v_7v_8 . \end{cases}$$

Expressing the obtained first integrals (43) by means of the original variables u_i and taking instead of D_7 the integral D_7/D_4, we obtain after a change of notation

$$(44)\quad \begin{cases} w_1 = u_1, \quad w_2 = u_3, \quad w_4 = u_9, \\ w_5 = u_2u_9 - u_3u_8 - u_4u_9 + u_6u_7 \\ w_6 = u_3u_6u_7 - u_1u_6u_9 + u_1u_8u_9 + u_2u_3u_9 - u_2u_7u_9 - u_3^2u_8 \\ w_7 = w = u_1u_5u_9 + u_2u_6u_7 + u_3u_4u_8 - u_3u_5u_7 - u_1u_6u_8 - u_2u_4u_9 . \end{cases}$$

From (29) it follows that

$$(45)\qquad w_7 = w = \det P, \quad P = (P_{ij}) \qquad (i,j = 1,2,3).$$

The general solution of (41) depends upon first integrals $w_1,\dots,w_7$, namely

$$(46)\qquad G = \psi(w_1,\dots,w_7),$$

where ψ is an arbitrary function of class C^1.

It is easy to verify that the first integral $w = w_7$ of the form (45) is a solution for all homogenous equations of the system (37), and the equation (37.1) is satisfied for

$r = -2$. We represent the matrix P in the form of a sum of symmetric and antisymmetric parts

$$(47) \qquad P = P_s + P_a .$$

It is easy to see that $\det P_a = 0$, but $w_s = \det P_s$ is a solution of (37) for $r = -2$. In particular, w_s is an integral of the equation (37.8). Hence the determinant of w_s is expressible in terms of first integrals of this equation. After a tedious computations we have succeeded in representing it in the following from (where in agreement with (39) and (40) we put $v_i = C_i$, $i = 1,\dots,9$).

$$(48) \qquad \begin{aligned} w_s = \det(P_s) = \frac{1}{4v_4}\Big[&-4v_1v_4v_6 - v_1v_4v_5^2 + v_2^2v_6 + v_3^2v_6 + \\ &+ v_2v_3v_5^2 + v_2v_5v_7 - v_3v_5v_7 + v_2v_5v_8 - v_3v_5v_8 + 2v_2v_3v_6 - 2v_7v_8 - v_7^2 - v_8^2\Big]. \end{aligned}$$

From (12), (29), (45), (47) it follows that the quantities w and w_s are W-densities of weight 2, i.e. defining w and w_s by formulas (45) and (48), respectively, we obtain the following transformation rules

$$(49) \qquad w' = |J|^{-2} w, \quad w_s' = |J|^{-2} w_s .$$

To simplify solving the system of equations (37) we take as first integrals of the equation (37.8) the following expressions:

$$(50) \qquad \begin{cases} r_1 = u_1,\ r_2 = u_3,\ r_3 = u_7,\ r_4 = u_9,\ r_5 = u_2u_9 - u_3u_8, \\ r_6 = u_4u_9 - u_6u_7,\ r_7 = w,\ r_8 = w_s . \end{cases}$$

Hence the general solution of (37.8) has the form

$$(51) \qquad G = \omega(r_1,\dots,r_8),$$

where ω is an arbitrary function of class C^1.

Next substituting G of the form (51) again into the equation (37.2) and solving the corresponding system of ordinary differential equations we obtain the following first integrals

$$(52)\qquad \begin{cases} s_1 = u_1, \quad s_2 = u_3, \quad s_3 = u_7, \quad s_4 = u_9, \\ s_5 = u_2u_9 - u_3u_8 - u_4u_9 + u_6u_7, \quad s_6 = w, \quad s_7 = w_s . \end{cases}$$

In view of (52) the general solution of (37.2) has the form

$$(53)\qquad G = \alpha(s_1, \dots, s_7),$$

where α is an arbitrary function of class C^1.

Substituting G of the form (53) into (37.7) and taking into account (52) we obtain the following partial differential equation

$$(54)\qquad (s_2 + s_3)\alpha_1 + s_4\alpha_2 + s_4\alpha_3 = 0,$$

which gives a system of ordinary differential equations

$$(55)\qquad \frac{ds_1}{s_2+s_3} = \frac{ds_2}{s_4} = \frac{ds_3}{s_4} = \frac{ds_4}{0} = \frac{ds_5}{0} = \frac{ds_6}{0} = \frac{ds_7}{0}.$$

This system has the following first integrals

$$(56)\qquad \begin{cases} t_1 = s_4 = u_9 \\ t_2 = s_5 = u_2u_9 - u_3u_8 - u_4u_9 + u_6u_7 , \\ t_3 = s_6 = w , \\ t_4 = s_7 = w_s , \\ t_5 = s_2 - s_3 = u_3 - u_7 , \\ t_6 = s_1s_4 - s_2s_3 = u_1u_9 - u_3u_7 . \end{cases}$$

Hence the general solution of (37.7) has the form

$$(57)\qquad G = \beta(t_1, \dots, t_6),$$

where β is an arbitrary function of class C^1.

Substituting G of the form (57) into (37.5) and taking into account (56) we obtain the following partial differential equation

(58) $$t_5\beta_5 + 2t_6\beta_6 = 0.$$

The equation (58) yields a system of ordinary differential equation

(59) $$\frac{dt_1}{0} = \frac{dt_2}{0} = \frac{dt_3}{0} = \frac{dt_4}{0} = \frac{dt_5}{t_5} = \frac{dt_6}{2t_6},$$

which has the following first integrals:

(60) $$\begin{cases} p_1 = t_1 = u_9, \quad p_2 = t_2 = u_2u_9-u_3u_8-u_4u_9+u_6u_7 , \\ p_3=t_3=w, \ p_4=t_4=w_s, \ p_5=t_5^2/t_6 = (u_3-u_7)^2(u_1u_9 - u_3u_7). \end{cases}$$

We see that the general solutions of (37.5) has the form

(61) $$G = \gamma(p_1,\dots,p_5),$$

where γ is an arbitrary function of class C^1.

Substituting (61) into (37.9) we obtain the equation

(62) $$2p_1\gamma_1 + p_2\gamma_2 = 0,$$

which yields a system of ordinary differential equations

(63) $$\frac{dp_1}{2p_1} = \frac{dp_2}{p_2} = \frac{dp_3}{0} = \frac{dp_4}{0} = \frac{dp_5}{0}.$$

This system has the following first integrals

(64) $$\begin{cases} q_1 = p_3 = w, \ q_2 = p_4 = w_s, \ q_3 = p_5 = \dfrac{(u_3-u_7)^2}{u_1u_9-u_3u_7} \\ q_4 = \dfrac{p_2^2}{p_1} = \dfrac{(u_2u_9-u_3u_8-u_4u_9+u_6u_7)^2}{u_9}. \end{cases}$$

Hence the general solution of (37.9) has the form

$$G = \delta(q_1, q_2, q_3, q_4), \tag{65}$$

where δ is any funtion of class C^1.
Substituting (65) into (37.3) we obtain

$$X_3(q_4)\delta_1 = 0, \tag{66}$$

where $X_3(q_4)$ is the value of the left-hand side of (37.3) for $G = q_4$. Because q_4 is not an integral of (37.3), we have

$$\delta_1 \equiv 0, \tag{67}$$

which means that the function $\delta(q_1, q_2, q_3, q_4)$ does not depend upon q_4. Let us observe that $X_3(q_4)$ has the form

$$X_3(q_4) = \frac{A \cdot B}{u_9^2}, \tag{68}$$

where

$$A = u_2u_9 - u_3u_8 - u_4u_9 + u_6u_7,$$

$$B = 2u_1u_6u_9 - 2u_1u_8u_9 + u_4u_7u_9 - u_2u_3u_9 + u_2u_7u_9 + u_3u_7u_8 - u_3u_4u_9 - u_3u_6u_7 + u_3^2u_8 - u_6u_7^2.$$

From (65) and (67) we finally obtain the following solution of (37.3)

$$G = \varkappa(q_1, q_2, q_3), \tag{69}$$

where $\varkappa$ is an arbitrary function of class C^1.
Substituting (69) into (37.4) we obtain

$$X_4(q_3)\varkappa_3 = 0, \tag{70}$$

where $X_4(q_3)$ is the value of the left-hand side of (37.4) for $G = q_3$. Since q_3 is not an integral of (37.4), we obtain

$$\varkappa_3 \equiv 0, \tag{71}$$

i.e. $\varkappa$ does not depend upon q_3.

Let us observe that

$$X_4(q_3) = \frac{(u_3 - u_7)B}{(u_1u_9 - u_3u_7)^2}. \tag{72}$$

In virtue of (69) and (71) the function G has the form

$$G = \varrho(q_1, q_2) = \varrho(w, w_s), \tag{73}$$

where ϱ is any function of class C^1.

Finally, substituting (73) into (37.1) we get

$$w\varrho_1 + w_s\varrho_2 = -\frac{r}{2}\varrho. \tag{74}$$

From (74) and (73) it follows that the conditions of Euler's theorem on homogenous functions holds, consequently G has the form

$$G = \omega(w, w_s), \quad w^2 + w_s^2 > 0, \tag{75}$$

where in the case of density concomitant $(r \neq 0)$ ω is an arbitrary function of class C^1 which is positively homogenous of order $(-\frac{r}{2})$, and in the case of scalar concomitants ω is an arbitrary function of class C^1 homogenous of zero order.

It is not difficult to verify that G of the form (75) satisfies the equation (37.6).

Taking into account Euler's theorem on homogenous functions and the fact that the sought concomitant must satisfy the functional equation (10) (for $\varepsilon = 1$ with the conditions

(29) - (34)), we infer from (75) and (49) that the function G can be represented in the form

$$G = |w|^{-r/2} K(w_s/w), \quad w \neq 0, \tag{76}$$

where K is any function of class C^1.

Hence by (49) the object

$$\sigma = \frac{w_s}{w} \tag{77}$$

is a scalar.

Remark 1. The function G of the form (75) or (76) satisfies the original function equation (10) (with the conditions (29) - (34)) for $\varepsilon = 1$ only.

Remark 2. For $w = w_s = 0$ the equation (74) is satisfied for $r \neq 0$ by the function $G = 0$, and for $r = 0$ by the function $H \equiv C$, where C is any constant.

Hence we have proved the following theorems.

Theorem 2. In the projective-euclidean space A_3 every density concomitant $G(u_1,\dots,u_9)$ with weight $(-r)$ (in the class C^1) of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ is a W-density of the form

$$G = (w, w_s), \quad w^2 + w_s^2 > 0, \quad r \neq 0, \tag{78}$$

where ω is any function of class C^1 which is positively homogenous of order $(-\frac{r}{2})$.

Theorem 3. In the projective-euclidean space A_3 every scalar concomitant $G(u_1,\dots,u_9)$ (in the class C^1) of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ has the form

$$G = \tau(w, w_s), \quad w^2 + w_s^2 > 0, \tag{79}$$

where τ is any function of class C^1 which homogenous of zero order.

The densities w and w_1 appearing in Theorem 2 and 3 are defined by formulas (45) and (48), respectively.

C o r o l l a r y 2. In the projective-euclidean space A_3 (in the class C^1) there exists no non-trivial algebraic concomitant of the curvature tensor $R_{\alpha\beta\gamma}{}^{\delta}$ which is a G-density.

R e m a r k 3. When the function ω in (75) is linear, we have

(80) $$G = aw + bw_s ,$$

where a and b are any constant. In fact, in view of (49) the function G of the form (80) is a W-density of weight 2. Assuming that the sought density concomitant G is of the form

(81) $$G = a^{ijk}u_i u_j u_k, \qquad (i,j,k = 1,\dots,9)$$

and substituting it into the system of equations (37), we obtain, after a tedious computation, the form (80) for the function G.

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