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ON SOME RIEMANNIAN MANIFOLDS ADMITTING A CONCIRCULAR VECTOR FIELD

1. Introduction

Let M_n $(n > 2)$ be an n-dimensional Riemannian manifold of class C^∞. By $R^h{}_{ijk}$, R_{ij} and R we denote the curvature tensor, Ricci tensor and the scalar curvature respectivity.

A non zero vector field v^h satisfying the condition

$$(1) \qquad v^h{}_{,i} = C\,\delta^h_i + v^h B_i$$

is called a concircular vector field and if $B_i = 0$ then v^h is called a special concircular vector field [3], where B and C are some scalar fields, $B_i = B_{,i}$ and the comma denotes covariant differantiation with respect to the metric of M_n.

This paper is concerned with some generalizations of theorems proved by T.Koyanagi in [1].

Moreover, we shall consider a Riemannian manifold M_n $(n > 2)$ which has the Ricci tensor satisfying

$$(2) \qquad R_{ij,k} + R_{jk,i} + R_{ki,j} = 0$$

and admits a concircular vector field (see [4]).

2. Preliminary results

L e m m a 1. If the manifold M_n admits a concircular vector field v^h, then the following conditions hold:

(3) $$C\cdot R_{khij} + v_s R^s{}_{hij,k} = T_{jk}g_{hi} - T_{ik}g_{hj},$$

(4) $$C\cdot R_{kj} + v_s R^s{}_{j,k} = (n-1)T_{jk},$$

(5) $$C\cdot R_{hi} + v^s R_{hi,s} - v^s R_{is,h} = g^{rs}T_{rs}\, g_{hi} - T_{ih},$$

where

(6) $$T_{jk} = D_j B_k - D_{j,k},$$

(7) $$D_j = C_j - C\cdot B_j$$

and

$$C_j = C_{,j}\ .$$

P r o o f. From (1) we have

(8) $$v_{h,i} = C\cdot g_{hi} + v_h B_i.$$

Differentiating (8) covariantly and taking into account (8) again we find

$$v_{h,i,j} = C_j g_{hi} + C\cdot g_{hj}B_i + v_h B_j B_i + v_h B_{i,j}.$$

Hence, by means of the Ricci identity, in view of (1) and (7), we obtain

(9) $$- v_s R^s{}_{hij} = D_j g_{hi} - D_i g_{hj}.$$

By transvection with v^h the last equation gives

(10) $$D_j v_i = D_i v_j.$$

Differentiating (9) covariantly and using (8) and (9) we find

$$-C\cdot R_{khij} + B_k(D_j g_{hi} - D_i g_{hj}) - v_s R^s{}_{hij,k} = D_{j,k} g_{hi} - D_{i,k} g_{hj}$$

which, in view of (6), leads immediately to (3).

Furthermore, contracting (3) with g^{hi} we easily obtain (4).

Contracting (3) with g^{jk}, we find

$$C\cdot R_{hi} + v^s R^r{}_{ihs,r} = g^{rs} T_{rs} g_{hi} - T_{ih},$$

which, by making use of the well-known formula

$$R^r{}_{ihs,r} = R_{ih,s} - R_{is,h}$$

leads to (5). Our lemma is thus proved.

L e m m a 2. Let M_n admit a concircular vector field v^h.

If for M_n the equation (2) is satisfied, then the condition

$$(11) \qquad v^r v^s R_{ks,r} = v^r v^s R_{rs,k} = 0$$

holds.

P r o o f. Contracting (4) with g^{jk} and making use of the well-known formula

$$\frac{1}{2} R_{,k} = R^r{}_{k,r}$$

we get

$$C\cdot R + \frac{1}{2} v^s R_{,s} = (n-1) g^{rs} T_{rs}.$$

Since the scalar curvature R of the M_n is constant, we have

$$C\cdot R = (n-1) g^{rs} T_{rs},$$

whence, by virtue of (5), we obtain

$$(12)\qquad C.R_{hi} + v^sR_{hi,s} - v^sR_{is,h} = \frac{R}{n-1}C\cdot g_{hi} - T_{ih}\ .$$

But (12), together with (4), yields

$$2CR_{hi} + v^sR_{hi,s} - (n-2)T_{ih} = \frac{R}{n-1}C\cdot g_{hi}\ ,$$

which implies that the tensor T_{ih} is symmetric. Interchanging in (12) the indices h and i we obtain

$$C\ R_{ih} + v^sR_{ih,s} - v^sR_{hs,i} = \frac{R}{n-1}C\cdot g_{ih}\cdot T_{hi}$$

so,by (12) and symmetry of the tensor T_{hi}, we have

$$(13)\qquad v^sR_{sh,i} = v^sR_{si,h}$$

and

$$(14)\qquad v^rv^sR_{sh,r} = v^rv^sR_{sr,h}.$$

Transvecting (2) with v^j and substituting (13), we get

$$(15)\qquad - v^sR_{ik,s} = 2\ v^sR_{is,k}.$$

But the last relation gives

$$- v^sv^rR_{rk,s} = 2\ v^sv^rR_{rs,k}$$

which, in view of (14), completes the proof.

L e m m a 3. Let M_n be a connected analytic manifold admitting a concircular vector field v^h. If the condition (2) is satisfied and C is non-zero, then

$$(16)\qquad T_{jk} = \frac{R}{n(n-1)}C\cdot g_{jk}.$$

P r o o f. Transvecting the relations (4) with v^j and (12) with v^h and using (11) we obtain

$$C\cdot v^s R_{ks} - (n-1)v^s T_{ks} = 0$$

and

$$C\cdot v^s R_{ks} + v^s T_{ks} = \frac{R}{n-1}\, C\cdot v_k$$

respectively, hence

$$C\cdot v^s R_{sk} = \frac{R}{n}\, C\cdot v_k$$

and consequently

(17) $$v^s R_{sk} = \frac{R}{n}\, v_k.$$

On the other hand, contracting (9) with g^{hi} we get

(18) $$- v^s R_{sj} = (n-1)D_j.$$

Substituting the last equation into (17) we have

(19) $$(n-1)D_j = -\frac{R}{n}\, v_j.$$

Differentiating now (19) covariantly and using (8), (19) and (6) we obtain (16). Thus the lemma is proved.

3. Main results

T h e o r e m 1. Let M_n $(n > 2)$ be a connected and analytic Riemannian manifold. If M_n admits a non-zero concircular field, D_h is non-zero, and the condition

(20) $$R_{hi,j,k} - R_{hi,j,k} = 0$$

is satisfied, then M_n is an Einstein space.

P r o o f. Differentiating (4) covariantly and applying (4) again, we get

$$C_hR_{kj} + C\cdot R_{kj,h} + C\cdot R_{hj,k} + (n-1)B_hT_{jk} - C\cdot B_h\cdot R_{kj} + \tag{21}$$
$$+ v_sR^s{}_{j,k,h} = (n-1)T_{jk,h}.$$

Interchanging now the indices h,k, in (21) and making use of relations (7) and (20), we find

$$D_hR_{kj} - D_kR_{hj} = (n-1)(T_{jk,h}-T_{jh,k}-B_hT_{jk}+B_kT_{jh}). \tag{22}$$

On the other hand, as an immediate consequence of (6), and (7), we get

$$D_{j,h,k} - D_{j,k,h} = T_{jk,h} - T_{jh,k} - B_hT_{jk} + B_kT_{jh}$$

whence, making use of the Ricci identity, we have

$$T_{jk,h} - T_{jh,k} - B_hT_{jk} + B_kT_{jh} = - D_sR^s{}_{jhk}.$$

Substituting the last equation into (22) we obtain

$$D_hR_{kj} - D_kR_{hj} = -(n-1)D_sR^s{}_{jhk}.$$

Multiplying both sides of this equation by v_i and using (10) and (9) we obtain

$$D_i\left[v_hR_{kj} - v_kR_{hj} - (n-1)\ (D_kg_{jh} - D_hg_{jk})\right] = 0,$$

so

$$v_hR_{kj} - v_kR_{hj} = (n-1)(D_kg_{jh} - D_hg_{jk}). \tag{23}$$

Contracting now (23) with g^{kj} and making use of (18) we find

$$D_h = -\frac{R}{n(n-1)}v_h. \tag{24}$$

The last equation, together with (23), gives

$$v_h R_{kj} - v_k R_{hj} = -\frac{R}{n}(v_k g_{jh} - v_h g_{jk}).$$

By transvection with v^h, this yields, in view of (18) and (24),

$$v^s v_s (R_{kj} - \frac{R}{n} g_{kj}) = 0$$

and

$$R_{kj} = \frac{R}{n} g_{kj}.$$

Our theorem is thus proved.

From Theorem 1 we conclude the following corollary.

Corollary 1. Let M_n $(n > 2)$ be a connected and analytic manifold. If M_n admits a spécial concircular vector field, with non-constant function C, and the condition (20) is satisfied, then M_n is an Einstein space.

Theorem 2. Let M_n $(n > 2)$ be a connected and analytic manifold. If M_n admits a non-zero concircular vector field, D_h is non-zero, and the condition.

(25) $$R_{hijk,l,m} - R_{hijk,m,l} = 0$$

is satisfied, then M_n is a space of constant curvature.

Proof. From Theorem 1 and (20) it follows that M_n is an Einstein space. Therefore the equation (4) can be written as

$$T_{jk} = \frac{R}{n(n-1)} C \cdot g_{jk}.$$

Substituting now the last relation into (3) we get

$$C \cdot R_{khij} + v_s R^s{}_{hij,k} = \frac{R}{n(n-1)} C(g_{jk} g_{hi} - g_{ik} g_{hj}).$$

By covariant differentiation, this formula yields in view of (8) and (7)

$$D_l R_{khij} + C \cdot R_{khij,l} + C \cdot R_{lhij,k} + v_s R^s{}_{hij,k,l} =$$

$$= \frac{R}{n(n-1)} D_l (g_{jk} g_{hi} - g_{ik} g_{hj}).$$

Interchanging the indices k and l in the last equation and applying (25) we find

$$D_l R_{khij} - D_k R_{lhij} = \frac{R}{n(n-1)} \Big[D_l (g_{jk} g_{hi} - g_{ik} g_{hj}) -$$

$$- D_k (g_{jl} g_{hi} - g_{il} g_{hj}) \Big]$$

from which, in the same way as in the proof of Theorem 1, we obtain

$$v_l R_{khij} - v_k R_{lhij} = \frac{R}{n(n-1)} \Big[v_l (g_{jk} g_{hi} - g_{ik} g_{hj}) -$$

$$- v_k (g_{jl} g_{hi} - g_{il} g_{kj}) \Big].$$

Transvecting the last equation with v^l and making use of (9) and (19), we find

$$R_{khij} = \frac{R}{n(n-1)} (g_{jk} g_{hi} - g_{ik} g_{hj}),$$

which, evidently, completes the proof.

From Theorem 2 we easily obtain

C o r o l l a r y 2. Let M_n $(n>2)$ be a connected and analytic manifold. If M_n admits a special concircular vector field, with non-constant function C, and the condition (25) is satisfied, then M_n is a space of constant curvature.

T h e o r e m 3. Let M_n $(n>2)$ be a connected and analytic manifold admitting a concircular vector field. If the condition (2) is satisfied and C is non-zero, then M_n is an Einstein space.

P r o o f. Substituting (15) into (12) we get

$$C\cdot R_{hi} - 3\, v^s R_{is,h} + T_{ih} = \frac{R}{n-1} C\cdot g_{ih},$$

which together with (4) yields

$$4C\cdot R_{hi} - (3n-4)T_{hi} = \frac{R}{n-1} C\cdot g_{hi}$$

But in view of (16) the last equation gives

$$C(R_{hi} - \frac{R}{n} g_{hi}) = 0,$$

which completes the proof.

Since every Ricci symmetric manifold M_n satisfied the condition (2), we have

C o r o l l a r y 3. Every connected and analytic Ricci symmetric manifold M_n admitting a concircular vector field, such that the scalar function C is non-zero, is an Einstein space (see [2]).

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