DEMONSTRATIO MATHEMATICA
Vol. IX No 3 1976

Tomasz Marks

TAYLOR EXPANSION IN THE FINITE ELEMENT METHOD FOR A TWO-POINT BOUNDARY VALUE PROBLEM

Introduction

The finite element method has become very important today as one of the most powerful tools for finding numerical solutions of boundary value problems.

The aim of this paper is to show the application of the Taylor expansion formula to specify admissible functions in the case of a two-point boundary value problem. Sections 1 and 2 give a short introduction to the problem discussed. A simple example of piecewise-linear approximations is described in Section 3. The Taylor expansion approach is demonstrated in Section 4. Because of the practical point of view there is no theoretical error analysis here. But in Section 5 the solution of a particular problem is shown and a comparison with piecewise linear and analytical solutions is made.

1. Statement of the problem

Many technical problems are expressed in the form of a two-point boundary value problem of the form:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \tag{1.1}$$

$$u(a) = u_0;\ u(b) = u_n, \tag{1.2}$$

over the closed interval $I=[a,b]$; $p(x)$, $q(x)$, $f(x)$ being given possibly nonlinear functions of a real variable x; u_0, u_n - constants. For the sake of simplicity the form (1.2) of boundary conditions has been chosen, but it should be noted that more general form might be discussed as well.

It can be shown - under appropriate assumptions concerning $p(x), q(x), f(x)$ - that if problem (1.1), (1.2) has a solution, then this solution will be unique. It can also be demonstrated that this problem is then equivalent to one of finding a minimum of the functional

$$F(u) = \int_a^b \left[p(x)\left(\frac{du}{dx}\right)^2 + q(x)u^2 - 2f(x)u\right]dx \tag{1.3}$$

over a set of functions that satisfy the conditions (1.2) (see [1] for necessary assumptions).

2. Finite element method

The finite element method approach requires the interval I to be divided into a finite number of subintervals $I_i = [x_{i-1}, x_i]$. Let us assume that this division is given by n+1 nodal points

$$a = x_0 < x_1 < \ldots\ldots < x_{n-1} < x_n = b.$$

For simplicity let us take the subintervals being of the same length, i.e.

$$x_i - x_{i-1} = h \qquad i=1,2,\ldots,n, \qquad \text{where } h = \frac{b-a}{n}.$$

Under appropriate assumptions (see [1]) the functional F breaks up into the sum of functionals $F_1, F_2, \ldots, F_n$, i.e.

$$F(u) = \sum_{i=1}^{n} F_i(u) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} H(u)dx, \tag{2.1}$$

where $H(u)=p(x)\left(\frac{du}{dx}\right)^2 + q(x)u^2 - 2f(x)u$.

The finite element method is the generalized Ritz method working with special admissible functions which are piecewise defined. Here, these functions are considered which have different analytical expressions over each subinterval. In our one-dimensional case, a finite element is a closed subinterval I_i with the family of functions which are allowed to occur within it. This family is a linear combination with coefficients p_{ij} of a finite number of chosen functions $\varphi_{ij}(x)$, so that

$$(2.2) \qquad \varphi^i(x)=p_{i1}\,\varphi_{i1}(x)+\dots+p_{ik}\,\varphi_{ik}(x).$$

Now, the problem is to find such a spline-function of the form

$$(2.3) \qquad \varphi(x) = \begin{cases} \varphi^1(x) & \text{for } x \in I_1, \\ \vdots & \\ \varphi^n(x) & \text{for } x \in I_n, \end{cases}$$

where φ^i is given by (2.2), which is minimizing the functional F and fulfiling boundary conditions (1.2). In other words, the approximate solution is determined by making the functional F stationary in the set of all functions (2.3) which satisfy the conditions (1.2) and have the component functions φ^i with the same values at the coincident nodes of the adjacent elements (the last requirement will be recalled as the continuity conditions). This will lead to the system of linear equations with respect to unknown values of the parameters p_{ij}.

The existence, uniqueness and convergence, when h tends to zero, of the approximate solutions can be obtained from the Lax-Milgram Theorem (see [1] Chapter 1).

3. Piecewise linear approximation

The admissible functions can be given directly in terms of their own nodal values. Let us denote

$$y_k = \varphi(x_k) \qquad \text{for } k=0,1,\dots,n,$$

where $\varphi(x)$ is given by (2.3) and (2.2). Then, a piecewise linear approximation is defined over i-th subinterval by

$$(3.1) \qquad \varphi^i(x) = \frac{x_i-x}{h} y_{i-1} + \frac{x-x_{i-1}}{h} y_i \quad i=1,2,\dots,n,$$

y_i are chosen as parameters. It should be noted here that the continuity conditions hold. Thus, we obtain

$$(3.2) \qquad F_i(\varphi) = F_i(\varphi^i) = \int_{x_{i-1}}^{x_i} \Big[p(x) \left(\frac{y_i-y_{i-1}}{h}\right)^2 + q(x)\left(\frac{x_i-x}{h} y_{i-1} + \frac{x-x_{i-1}}{h} y_i\right)^2 - 2f(x)\left(\frac{x_i-x}{h} y_{i-1} + \frac{x-x_{i-1}}{h} y_i\right)\Big] dx$$

for $i=1,2,\dots,n$. It is easy to see that

$$(3.3) \qquad \frac{\partial F_i(\varphi^i)}{\partial y_j} = 0 \qquad \text{for } j \neq i-1, i .$$

Furthermore

$$(3.4) \qquad \frac{\partial F(\varphi)}{\partial y_j} = \frac{\partial F_j(\varphi)}{\partial y_j} + \frac{\partial F_{j+1}(\varphi)}{\partial y_j} \quad \text{for } j=1,\dots,n-1.$$

There is no difficulty in verifying that

$$(3.5) \qquad \frac{\partial F_i(\varphi)}{\partial y_i} = a_i y_{i-1} + b_i y_i - e_i, \qquad \frac{\partial F_i(\varphi)}{\partial y_{i-1}} = c_i y_{i-1} + a_i y_i - f_i,$$

$(i=1,2,\dots,n-1)$, where a_i, b_i, c_i, e_i, f_i can be obtained by analytical or even numerical integration. Introducing (3.5) to (3.4) yields

$$(3.6) \qquad \frac{\partial F(\varphi)}{\partial y_i} = a_i y_{i-1} + (b_i + c_{i+1}) y_i + a_{i+1} y_{i+1} - e_i - f_{i+1} \qquad (i=1,2,\dots,n-1).$$

Taking into considerations (1.2), the stationary conditions $\frac{\partial F(\varphi)}{\partial y_i} = 0$ $(i=1,2,\dots,n-1)$ lead to the system of n-1 linear equations

$$\begin{aligned}(b_1 + c_2)y_1 + a_2y_2 &= e_1+f_2-a_1u_0,\\ a_iy_{i-1} +(b_i+c_{i+1})y_i+a_{i+1}y_{i+1} &= e_i + f_{i+1},\\ &\quad (i=2,3,\dots,n-2),\\ a_{n-1}y_{n-2} + (b_{n-1} + c_n)y_{n-1} &= e_{n-1} + f_n - a_nu_n,\end{aligned} \tag{3.7}$$

with a tridiagonal and symmetric matrix. The solution of (3.7) gives directly the values of the approximate function in nodal points which in turn can be used to obtain analytical expressions for φ^i.

4. Taylor expansion in the finite element method

Let us assume that $p(x) \neq 0$ for $x \in I$. Thus, equation (1.1) can be rewritten in the form

$$\frac{d^2u}{dx^2} = r(x)\,\frac{du}{dx} + s(x)u + t(x), \tag{4.1}$$

where

$$r(x) = \frac{p'(x)}{p(x)}, \quad s(x) = -\frac{q(x)}{p(x)}, \quad t(x) = -\frac{f(x)}{p(x)}.$$

Now, let r(x), s(x), t(x) and the solution of (4.1), (1.2) be infinitely differentiable functions on I (regularity assumptions). We use the Taylor expansion formula to express u on each subinterval I_i in the following form

$$u(x) = u(z_i)+u'(z_i)(x - z_i + \frac{1}{2}\,u''(z_i)(x - z_i)^2 + \dots, \tag{4.2}$$

where $z_i=x_{i-1} + \frac{h}{2}$. Denoting

$$\begin{aligned} u(z_i) &= u_i,\\ u'(z_i) &= v_i,\end{aligned} \tag{4.3}$$

for $i=1,2,\dots,n$, we deduce from (4.1) and from the regularity assumptions that

$$
(4.4)\qquad
\begin{aligned}
u^{(2)}(z_i) &= r_i v_i + s_i u_i + t_i,\\
u^{(3)}(z_i) &= (r_i+r_i'+s_i)v_i+(r_i s_i + s_i')u_i+(r_i t_i + t_i'),\\
u^{(4)}(z_i) &= \dots\dots,\\
&\dots\dots\dots\dots\dots,\\
&\dots\dots\dots\dots\dots,
\end{aligned}
$$

where $r_i = r(z_i)$, $r_i' = r'(z_i)$, $s_i = s(z_i)$ and so on. Generally

$$
(4.5)\qquad u^{(k)}(z_i) = c_{ik}u_i + d_{ik}v_i + e_{ik},
$$

where the constant coefficients c_{ik}, d_{ik}, e_{ik} depend only on the values of the given functions $r(x)$, $s(x)$, $t(x)$ and on the values of their derivatives at the point z_i.

Now the piecewise-polynomial approximation can be chosen over i-th subinterval such that

$$
(4.6)\qquad
\begin{aligned}
\psi^i(x) &= u_i+v_i(x-z_i) + \sum_{k=2}^{m} \frac{1}{k!}(c_{ik}u_i+d_{ik}v_i+e_{ik})(x-z_i)^k =\\
&= a^i(x)u_i + b^i(x)v_i + c^i(x),
\end{aligned}
$$

where $a^i(x), b^i(x)$, $c^i(x)$ are polynomials of degree m. (It is appropriate to note here, that we can weaken our regularity assumptions, because only a finite number of the derivatives is needed for our purpose). Denoting

$$
\psi(x_i) = y_i \qquad \text{for } i=0,1,\dots,n,
$$

we have (for the continuity conditions to be fulfilled)

$$
(4.7)\qquad
\begin{aligned}
\psi^i(x_{i-1}) &= y_{i-1} = a^i(x_{i-1})u_i + b^i(x_{i-1})v_i + c^i(x_{i-1}),\\
\psi^i(x_i) &= y_i = a^i(x_i)u_i + b^i(x_i)v_i + c^i(x_i).
\end{aligned}
$$

The solution with respect to u_i, v_i yields

$$(4.8) \qquad u_i = A_1^i y_{i-1} + B_1^i y_i + C_1^i \;, \; v_i = A_2^i y_{i-1} + B_2^i y_i + C_2^i \;.$$

Introducing (4.8) to (4.6) we obtain for i=1,2,..,n

$$(4.9) \qquad \psi^i(x) = A^i(x) y_{i-1} + B^i(x) y_i + C^i(x),$$

where $A^i(x), B^i(x), C^i(x)$ are polynomials of degree m. If we choose y_i as parameters, all considerations from Section 3 can be easly repeated. As previously, the latter operations will lead to the system of linear equations with tridiagonal symmetric matrix. But now we have the approximate function consisting of higher degree polynomials whose form is obtained directly from the form of our differential equation. Therefore, we can expect that error will be less than in the previous case.

5. Example

The example has been chosen as simple as possible to enable a straightforward error veryfication of the method applied.

Let us consider a two-point boundary value problem

$$-\frac{d}{dx}\left(x\,\frac{du}{dx}\right) = 0, \qquad u(0.25) = 0.62, \qquad u(1.00) = 0.00.$$

The interval $[0.25, 1.00]$ has been devided into n subintervals $[x_{i-1}, x_i]$ with expansion points $z_i = x_{i-1} + \frac{h}{2}$, where $h = x_i - x_{i-1} = \frac{0.75}{n}$. Only first three components of the Taylor expansion have been taken into consideration. Thus, in the case presented, we have admissible functions which are piecewise defined as below

$$\psi(x) = \psi^i(x) = u_i + v_i(x - z_i) + \frac{1}{2z_i} v_i (x - z_i)^2 =$$

$$= u_i + \left[(x - z_i) + \frac{(x - z_i)^2}{2z_i}\right] v_i, \quad \text{for } x \in \left[x_{i-1}, x_i\right].$$

Tablica 1

n= 40, h= 0.01875

x	U1	U2	Y
0.2500	0.620 000 000	0.620 000 000	0.620 000 000
0.2688	0.587 664 606	0.587 655 639	0.587 655 635
0.2875	0.557 509 126	0.557 493 509	0.557 493 503
0.3063	0.529 258 202	0.529 237 666	0.529 237 658
0.3250	0.502 685 552	0.502 661 406	0.502 661 397
0.3438	0.477 602 956	0.477 576 208	0.477 576 198
0.3625	0.453 852 180	0.453 823 611	0.453 823 601
0.3813	0.431 298 922	0.431 269 145	0.431 269 135
0.4000	0.409 828 220	0.409 797 719	0.409 797 709
0.4188	0.389 340 909	0.389 310 070	0.389 310 060
0.4375	0.369 750 853	0.369 719 984	0.369 719 974
0.4563	0.350 982 757	0.350 952 106	0.350 952 096
0.4750	0.332 970 423	0.332 940 189	0.332 940 180
0.4938	0.315 655 341	0.315 625 686	0.315 625 677
0.5125	0.298 965 542	0.298 956 596	0.298 956 588
0.5312	0.282 914 657	0.282 886 527	0.282 886 519
0.5500	0.267 401 144	0.267 373 915	0.267 373 908
0.5688	0.252 407 636	0.252 381 378	0.252 381 371
0.5875	0.237 900 407	0.237 875 172	0.237 875 165
0.6063	0.223 841 900	0.223 824 735	0.223 824 728
0.6250	0.210 225 359	0.210 202 297	0.210 202 291
0.6437	0.197 004 483	0.196 982 552	0.196 982 546
0.6625	0.184 163 154	0.184 142 374	0.184 142 369
0.6813	0.171 680 188	0.171 660 574	0.171 660 569
0.7000	0.159 536 126	0.159 517 688	0.159 517 684
0.7187	0.147 713 052	0.147 695 798	0.147 695 794
0.7375	0.136 194 436	0.136 178 368	0.136 178 364
0.7563	0.124 964 989	0.124 950 110	0.124 950 106
0.7750	0.114 010 549	0.113 996 857	0.113 996 853
0.7938	0.103 317 969	0.103 305 460	0.103 305 457
0.8125	0.092 875 021	0.092 863 690	0.092 863 687
0.8312	0.082 670 314	0.082 660 157	0.082 660 154
0.8500	0.072 693 223	0.072 684 231	0.072 684 229
0.8688	0.062 933 813	0.062 925 979	0.062 925 977
0.8875	0.053 382 789	0.053 376 104	0.053 376 102
0.9062	0.044 031 438	0.044 025 893	0.044 025 892
0.9250	0.034 871 582	0.034 867 167	0.034 867 166
0.9438	0.025 895 536	0.025 892 241	0.025 892 240
0.9625	0.017 096 068	0.017 093 882	0.017 093 882
0.9813	0.008 456 365	0.008 465 278	0.008 465 277
1.0000	0.000 000 000	0.000 000 000	0.000 000 000

The method described in Section 4 brings the system of linear equations to be solved

$$(a_1 + a_2)y_1 - a_2y_2 = 0.62a_1,$$

$$-a_iy_{i-1}+(a_i+a_{i+1})y_i-a_{i+1}y_{i+1} = 0 \quad (i = 2,3,\dots,n-2),$$

$$-a_{n-1}y_{n-2} + (a_{n-1} + a_n)y_{n-1} = 0,$$

where

$$a_i = \frac{2}{h^2 z_i} \int_{x_{i-1}}^{x_i} x^3 dx, \qquad (i = 1,2,\dots,n).$$

Tab.1 gives a comparison of numerical solutions obtained by using the piecewise linear approximation (U1) - 4 exact digits, by using the piecewise Taylor expansion (U2) - 7 exact digits, and by the analytical solution (Y).

Of course, this comparison cannot replace the theoretical error analysis, but, it shows that the Taylor expansion approach is very useful.

REFERENCES

[1] J.P. Aubin: Approximation of elliptic boundary value problems. New York 1972.

[2] W.G. Bickley: Piecewise cubic interpolation and two-point boundary problems, Comput.J. 11(1968) 206-208.

[3] D.J. Fyfe: The use of cubic splines in the solution of two-point boundary value problems, Comput. J. 12(1969) 188-192.

[4] R. Kossowski: Application of the finite element method for solving linear differential equation with boundary conditions. Finite element method in flow problems. Swansea Huntswill 1974.

[5] G. Micula: Approximate solution of the differential equation $y'' = f(x,y)$ with spline functions, Math. Comp. 27(1973) 807-816.

[6] E.R. de Arantes e Oliveira: Theoretical foundations of the finite element method, Internat. J.Solids and Structures. 4(1968) 929-952.

[7] O.C. Z i e n k i e w i c z: Metoda elementów skończonych. Warszawa 1972.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

Received October 31, 1975.