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APPLICATION OF THE GREEN FUNCTION IN THE THEORY OF PSEUDOPARABOLIC EQUATIONS

We consider equation

$$(1)\quad \alpha[u] \equiv \frac{\partial^2}{\partial x \partial t}\left[p(x,t)\,\frac{\partial u}{\partial x}\right] + q(x,t)\,\frac{\partial u}{\partial t} = F(x,t,u,u_x,u_{xx},u_t,u_{xt})$$

with homogeneous boundary conditions

$$(2)\quad \begin{cases} \alpha_1 u(a,t) + \alpha_2 u_x(a,t) = 0, \\ \beta_1 u(b,t) + \beta_2 u_x(b,t) = 0, \\ \quad u(x,0) = 0, \end{cases}$$

where $(x,t) \in Q = \langle a,b\rangle \times \langle 0,T\rangle$, $(|\alpha_1| + |\alpha_2|)(|\beta_1| + |\beta_2|) \neq 0$.

We suppose that

1^0 $F(x,t,z_1 z_2,z_3,z_4,z_5) \in C(Q \times R^5)$,

2^0 $|F(x,t,z_1,z_2,z_3,z_4,z_5) - F(x,t,\tilde{z}_1,\tilde{z}_2,\tilde{z}_3,\tilde{z}_4,\tilde{z}_5)| \leqslant$

$$\leqslant L \sum_{i=1}^{5} |z_i - \tilde{z}_i|,$$

3^0 $p(x,t)$, $q(x,t)$, $p_x(x,t)$, $p_t(x,t)$, $p_{xt}(x,t)$, $q_t(x,t) \in C(Q)$,

$$p(x,t) \neq 0 \quad \text{for} \quad (x,t) \in Q,$$

4^o For an arbitrary $t \in \langle 0,T\rangle$ the equation

$$l^t(y) = \frac{d}{dx}\left[p(x,t)y'\right] + q(x,t)y = 0,$$

with conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0,$$

has in the interval $\langle a,b\rangle$ the unique solution $y \equiv 0$.

We shall show that under these assumptions and some additional hypotheses the problem (1) - (2) has the unique solution $u(x,t)$ which is continuous in Q as well as the derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_t(x,t)$, $u_{xt}(x,t)$.

In the paper [1] N. Calistru has considered the case of $p(x,t)$ and $q(x,t)$ independent of the time variable t.

L e m m a 1. Under the assumptions stated above the Green function $G(x,\xi;t)$ for the operator l^t belongs to the class $C_t^1\langle 0,T\rangle$. The derivative $G_x(x,\xi;t)$ is continuous and differentiable with respect to t for $x > \xi$ and $x < \xi$; moreover, the derivative $G_{xt}(x,t;\xi)$ is continuous in each of the domains $x \leqslant \xi$ and $x \geqslant \xi$, thus it is bounded in the set $\langle a,b\rangle \times \langle a,b\rangle \times \langle 0,T\rangle$.

P r o o f. It is known (cf. [2], p.520) that if we put

$$G(x,\xi;t) = \begin{cases} y_1(x,t)y_2(\xi,t) & \text{for } x \leqslant \xi \\ y_2(x,t)y_1(\xi,t) & \text{for } x \geqslant \xi, \end{cases} \tag{3}$$

where $y_i(x,t)$ $(i=1,2)$ is a solution of the equation $l^t(y)=0$ satisfying conditions

$$\alpha_1 y_1(a,t) + \alpha_2 y_1'(a,t) = 0, \tag{4}$$

$$\beta_1 y_2(b,t) + \beta_2 y_2'(b,t) = 0, \tag{5}$$

and for fixed $t \in \langle 0,T \rangle$ we have

$$y_1(x,t)y_2'(x,t) - y_2(x,t)y_1'(x,t) = \frac{1}{p(x,t)}, \tag{6}$$

then

$$y(x,t) = -\int_a^b G(x,\xi;t)l^t[y(\xi;t)]\,d\xi \tag{7}$$

for an arbitrary function $y(x;t)$ satisfying the homogeneous boundary conditions (2).

We observe that for arbitrary $y_1(x;t)$, $y_2(x,t)$ satisfying the equation $l^t(y) = 0$ with conditions (4),(5), respectively, we have

$$\frac{d}{dx}\left\{p(x,t)\left[y_1(x,t)y_2'(x,t) - y_2(x,t)y_1'(x,t)\right]\right\} = 0,$$

thus

$$y_1(x,t)y_2'(x,t) - y_2(x,t)y_1'(x,t) = \frac{\psi(t)}{p(x,t)}, \tag{8}$$

where $\psi(t) \neq 0$ for $t \in \langle 0,T \rangle$ which follows from the assumption 4^o.

Now we shall show that the functions $y_1(x,t)$, $y_2(x,t)$ are continuous and differentiable with respect to t. We consider the problem

$$l^t(y_1) = 0, \quad \alpha_1 c_1 + \alpha_2 d_1 = 0, \tag{9}$$

where $c_1 = y_1(a,t)$, $d_1 = y_1'(a,t)$ are supposed to be constant.

By the substitution

$$\frac{d^2}{dx^2} y_1(x,t) = \varphi(x,t), \tag{10}$$

$$f(x,t) = \frac{-p_x(x,t)}{p(x,t)} d_1 - \frac{q(x,t)}{p(x,t)}\left[d_1(x-a) + c_1\right], \tag{11}$$

$$N_o(x,\xi;t) = \frac{p_x(x,t)}{p(x,t)} + \frac{q(x,t)}{p(x,t)}(x-\xi) \tag{12}$$

we reduce the problem (9) to the integral equation of Volterra type

$$(13)\qquad \varphi(x,t) = f(x,t) - \int_a^x N_0(x,\xi;t)\,\varphi(\xi;t)\,d\xi .$$

Its solution is given by the formula

$$(14)\qquad \varphi(x,t) = f(x,t) - \int_a^x \pi(x,\xi;t)\, f(\xi;t)\,d\xi ,$$

where

$$(15)\qquad \pi(x,\xi;t) = \sum_{n=0}^{\infty} (-1)^n N_n(x,\xi;t)$$

and

$$(16)\quad N_k(x,\xi;t) = \int_\xi^x N_{k-1}(x,s;t)\,N_0(s,\xi;t)\,ds\ , \quad (k=1,2,\ldots).$$

From the assumption 3^o it follows that the functions (11) and (12) are continuous and differentiable with respect to t. From (16) it follows that $N_k(x,\xi;t)$ is continuous and differentiable with respect to t for $k = 1,2,\ldots$ Therefore, under the assumptions stated above, there exists a constant D such that

$$\left|N_k(x,\xi;t)\right| \le D^{k+1}\frac{|a-b|^k}{k!}\ ,$$

$$\left|\frac{\partial N_k}{\partial t}(x,\xi;t)\right| \le D(2D)^k\,\frac{|a-b|^k}{k!}\,.$$

Thus the resolvent (15) is continuous with respect to t and the derivative $\frac{\partial \pi}{\partial t}(x,\xi;t)$ exists and is continuous. Therefore $\varphi(x,t)\in C_t^1 <0,T>$; since

$$y_1(x,t) = \int_a (x-\xi)\,\varphi(\xi,t)\,d\xi + d_1(x-a) + c_1\ ,$$

$$y_1'(x,t) = \int_a^x \varphi(\xi;t)\,d\xi + d_1,$$

we have shown that $y_1(x;t)\in C_t^1 <0,T>$. A similar argument shows that $y_2(x;t)\in C_t^1<0,T>$. Therefore $\psi(t)\in C_t^1<0,T>$, further, the Green function defined by (3) and the functions $\tilde{y}_1(x,t)=\frac{y_1(x,t)}{\psi(t)}$

and $\tilde{y}_2(x,t) = y_2(x,t)$ have the required property (6) and satisfy the thesis of Lemma 1.

C o r o l l a r y. Lemma 1 implies the existence of a finite constant

$$(17) \quad \Gamma = \sup_{\substack{x,\xi \in \langle a,b\rangle \\ t \in \langle 0,T\rangle}} \Big\{ |G(x,\xi;t)|, \ |G_x(x,\xi;t)|, \ |G_t(x,\xi;t)|, \ |G_{xt}(x,\xi;t)| \Big\}.$$

Taking into account (1) and (7) we construct the operator A defined by

$$(Aw)(x,t) = -\int_a^b G(x,\xi;t)\int_0^t \Big[\bar{F}(\xi,\tau,w(\xi,\tau)) + \frac{\partial q(\xi,\tau)}{\partial \tau} w(\xi,\tau)\Big] d\tau d\xi$$

which transforms the set $\sum$ of all functions $w(x,t)$ defined on Q and continuous with derivatives w_x, w_{xx}, w_t, w_{xt} into itself. It is easy to show that the solution od the problem (1) - (2) is a fixed point of the operator A. In the set $\sum$ we define the metric

$$\varrho_\lambda(u,v) = \sup_Q \Big\{ |u(x,t) - v(x,t)| e^{-\lambda t},$$

$$|u_x(x,t) - v_x(x,t)| e^{-\lambda t}, \ |u_{xx}(x,t) - v_{xx}(x,t)| e^{-\lambda t},$$

$$|u_t(x,t) - v_t(x,t)| e^{-\lambda t}, |u_{xt}(x,t) - v_{xt}(x,t)| e^{-\lambda t} \Big\},$$

where λ is an arbitrary positive constant. It is known that $(\Sigma, \varrho_\lambda)$ is a Banach space. Analogously, as it is done in [1] we can show that

$$\varrho_\lambda(Aw,Av) \leq \max\Big\{\frac{\Gamma(b-a)}{\lambda}, \frac{M}{\lambda}, \Gamma(b-a)(1+\frac{1}{\lambda})\Big\}(5L+R)\,\varrho_\lambda(w,v)$$

for $w,v \in \Sigma$, where M is a positive constant, $R = \sup_Q |q_t(x,t)|$ L is the Lipschitz constant from the assumption 2^0, Γ is a constant defined by (17). If $5L + R < \frac{1}{\Gamma(b-a)}$ and λ is sufficiently large, then from Banach's theorem on contracting mappings it follows that the problem (1) - (2) has the unique solution.

If in the right-hand side of the equation (1) the functions u_t and u_{xt} do not occur, then we obtain the estimation

$$\varrho_\lambda(Aw,Av) \leq \max\left\{\frac{\Gamma(b-a)}{\lambda}, \frac{M}{\lambda}\right\}(5L + R)\,\varrho_\lambda(w,v).$$

Then the problem has the unique solution without any additional hypotheses.

One can pose the problem of approximation of the exact solution u(x,t) of the problem (1) - (2) with the right-hand side independent od u_t and u_{xt} by the solutions of an ordinary differential equation.

We state the new problem

$$v_0(x) = 0$$

$$\text{(2.1)}\quad \frac{d}{dx}\left[p(x,t_k)\frac{d}{dx}(v_{k+1} - v_k) + q(x,t_k)(v_{k+1}-v_k\right) = $$

$$= h\,F^*(x,t_k,v_k',v_k'')$$

with conditions

$$\alpha_1 v_{k+1}(a) + \alpha_2 v_{k+1}(a) = 0,$$

$$\beta_1 v_{k+1}(b) + \beta_2 v_{k+1}(b) = 0,$$

where $F^*(x,t,z_1,z_2,z_3) = F(x,t,z_1,z_2,z_3) + q_t(x,t)z_1$ and $t_k = k\cdot h,\ h = \frac{T}{n},\quad k = 0,1,\ldots,n.$

One can prove the following theorem analogous to Theorem 2.1 in [1].

T h e o r e m. Let $\overline{v}(x,t)$ be the exact solution of the problem (1) - (2) (the right-hand side is independent of

v_t, v_{xt}). If $F^*(x,t,z_1,z_2,z_3)$ has the total derivative with respect to t which is bounded in $Q \times R^3$, then the differences

$$v_k(x) - \bar{v}(x,t_k), \quad v_k'(x) - \frac{\partial \bar{v}(x,t_k)}{\partial x}, \quad v_k''(x) - \frac{\partial^2 \bar{v}(x,t_k)}{\partial x^2}$$

converge uniformly to 0 with respect to x for $n \to \infty$.

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