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GENERALISED STRONG NÖRLUND SUMMABILITY

1. Introduction

In [2], a definition of generalised strong Nörlund summability was given and some multiplication theorems concerning strong Nörlund summability of product, which is more general than the Cauchy product of two sequences, were established.

In the present paper certain inclusion theorems have been established which answer questions such as: "If one generalised Nörlund method includes another, is the same true of the associated generalised strong Nörlund methods?" Some relations between generalised strong Nörlund, generalised absolute Nörlund and generalised Nörlund summability have been established which, in particular, yield some interesting relations between strong Riesz, absolute Riesz and Riesz summability. It is interesting to note that the results of [4] follow as particular cases of our results.

2. Preliminaries

Throughout this paper H and H_1 will denote positive constants which may not be the same at each occurence.

Given any sequence $\{p_n\}$, we write

$$p(z) = \sum_{n=0}^{\infty} p_n z^n$$

whenever the series on the right converges. We define the sequence $\{k_n\}$ of constants by means of the following formal identity

(2.1) $$k(z) = \frac{q(z)}{p(z)}, \quad k_{-1}=0.$$

As usual we say that the sequence $\{p_n\} \epsilon \mu$ if

$$p_0 = 1, \quad p_n > 0, \quad \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \text{ for } n>0.$$

Let $\{p_n\}$ and $\{\alpha_n\}$ be sequences of numbers, real or complex, and write

$$(p*\alpha)_n = \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu .$$

It is well known that the operation $*$ is commutative and associative.

Given any arbitrary sequence $\{w_n\}$, we define

$$\Delta w_n = w_n - w_{n-1}, \quad w_{-1} = 0.$$

It can be easily verified that

$$\Delta(p*\alpha)_n = (\Delta p*\alpha)_n = (p*\Delta\alpha)_n.$$

<u>Definitions</u>

(1) <u>Generalised Nörlund summability (N,p,α).</u>

Suppose that $(p*\alpha)_n \neq 0$ for $n \geqslant 0$. A given sequence $\{s_n\}$ is said to be summable (N,p,α) to the value s, if $t_n \to s$ as $n \to \infty$, where

(2.2) $$t_n = \frac{(p*\alpha s)_n}{(p*\alpha)_n}$$

and

$$(p*\alpha s)_n = \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu s_\nu .$$

This is denoted by $s_n \to s(N,p,\alpha)$ (cf. [3], [5], [6]).

The method (N,p,α) is said to be <u>regular</u> if it preserves limits for convergent sequences.

(2) <u>Generalised absolute Nörlund summability $|N,p,\alpha|_\lambda, \lambda > 0$.</u>

We shall say that $\{s_n\}$ is absolutely summable (N,p,α) with index $\lambda > 0$, or summable $|N,p,\alpha|_\lambda$, if

$$\sum_{\nu=1}^{\infty} \nu^{\lambda-1}|t_\nu - t_{\nu-1}| < \infty,$$

where t_n is defined by (2.2). When $\lambda = 1$, $|N,p,\alpha|_\lambda$ summability reduces to $|N,p,\alpha|$ summability (cf. [5]).

(3) <u>Generalised strong Nörlund summability $[N,p,\alpha]_\lambda, \lambda > 0$.</u>

Let $(p*\Delta\alpha)_n \neq 0$, $(p*\alpha)_n \neq 0$ for $n \geq 0$. A given sequence $\{s_n\}$ is said to be strongly summable (N,p,α) with index $\lambda > 0$ to s, or summable $[N,p,\alpha]_\lambda$ to s, if

$$\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n} (p*\Delta\alpha)_\nu \left| \frac{(p*\Delta(\alpha s))_\nu}{(p*\Delta\alpha)_\nu} - s \right|^\lambda = o(1) \ ;$$

and is denoted by $s_n \to s\ [N,p,\alpha]_\lambda$ (cf. [2]). As remarked in [2], this definition is of use only when $(p*\alpha)_n \to \infty$ as $n \to \infty$.

The choice $\alpha_n = 1$ for $n \geq 0$ leads (N,p,α) summability to (N,p) summability, $|N,p,\alpha|_\lambda$ summability to $|N,p|_\lambda$ summability and $[N,p,\alpha]_\lambda$ summability to $[N,p]_\lambda$ summability (cf. [4]).

When $p_n = 1$ for $n \geq 0$, (N,p,α) summability reduces to Riesz summability $(\bar{N},\alpha)$ (cf. [5]), $|N,p,\alpha|_\lambda$ to absolute Riesz summability $|\bar{N},\alpha|_\lambda$ and $[N,p,\alpha]_\lambda$ to strong Riesz summability $[\bar{N},\alpha]_\lambda$.

If P and Q are methods of summability, Q is said to include P (written $P \subseteq Q$) if every sequence summable P is also summable Q to the same sum. P and Q are said to be equivalent (written $P \equiv Q$) if each includes the other.

In the rest of the paper it is assumed that

$$(p*\alpha)_n \neq 0, \ (p*\Delta\alpha)_n \neq 0, \ (q*\alpha)_n \neq 0, \ (q*\Delta\alpha)_n \neq 0 \text{ for } n \geqslant 0.$$

3. The Lemmas

In this section we collect some lemmas which will be required in the proof of our theorems.

L e m m a 1. Let $\{p_n\} \in \mu$ and $q_n > 0$ for $n \geqslant 0$.

(i) If

$$\frac{p_n}{p_{n-1}} \leqslant \frac{q_n}{q_{n-1}} \quad \text{for} \quad n > 0, \tag{3.1}$$

then $k_0 > 0$, $k_n \geqslant 0$ for $n > 0$.

(ii) If

$$\frac{q_n}{q_{n-1}} \leqslant \frac{p_n}{p_{n-1}} \quad \text{for} \quad n > 0, \tag{3.2}$$

then $k_0 > 0$, $k_n \leqslant 0$ for $n > 0$.

The proofs of (i) and (ii) are respectively contained in the proofs of Theorem 23 of [7] and Theorem 3 of [4].

L e m m a 2. ([1] Lemma 2). Let $\{p_n\} \in \mu$, $\alpha_n > 0$, $q_n > 0$ for $n \geqslant 0$, $q_n = O(p_n)$ and (3.1) hold. Then, if (N, q, α) is regular, (N, p, α) is regular.

L e m m a 3 ([2] Lemma 1). If

$$\sum_{\nu=0}^{n} |(p*\Delta\alpha)_\nu| = O((p*\alpha)_n), \tag{3.3}$$

then $[N,p,\alpha]_\lambda \subseteq [N,p,\alpha]_\mu$ for $\lambda > \mu > 0$.

L e m m a 4. ([6] Lemma 1 with $\alpha_n = \beta_n$). Let $\alpha_n \neq 0$ for $n \geq 0$.

Then $(N,p,\alpha) \subseteq (N,q,\alpha)$ if and only if

$$(3.4) \qquad \left(|k| * |(p*\alpha)|\right)_n = O\left((q*\alpha)_n\right),$$

and, for every fixed ν,

$$(3.5) \qquad k_{n-\nu} = o((q*\alpha)_n).$$

L e m m a 5 ([1] Theorem 1). If $\{p_n\} \in \mu$, $\alpha_n > 0$, $q_n > 0$ for $n \geq 0$, $p_n = O(q_n)$, (N,q,α) is regular and (3.2) holds, then (N,p,α) is regular and $(N,p,\alpha) \subseteq (N,q,\alpha)$.

L e m m a 6 ([6] Theorem 1 with $\alpha_n = \beta_n$). If $\{p_n\} \in \mu$, $\alpha_n > 0$, $q_n > 0$ for $n \geq 0$, (3.1) holds and (N,q,α) is regular, then $(N,p,\alpha) \subseteq (N,q,\alpha)$.

L e m m a 7. Let $\alpha_n \neq 0$ for $n \geq 0$. Then $(\bar{N},\alpha) \subseteq (\bar{N},\beta)$ if and only if

$$(3.6) \qquad \sum_{\varrho=0}^{n} |d_{n,\varrho}| = O(1),$$

and, for every fixed ϱ,

$$(3.7) \qquad d_{n,\varrho} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$(3.8) \qquad d_{n,\varrho} = \begin{cases} \left(\dfrac{\beta_\varrho}{\alpha_\varrho} - \dfrac{\beta_{\varrho+1}}{\alpha_{\varrho+1}}\right) \dfrac{(1*\alpha)_\varrho}{(1*\beta)_n} & (0 \leq \varrho < n-1), \\[2ex] \dfrac{\beta_\varrho (1*\alpha)_\varrho}{\alpha_\varrho (1*\beta)_n} & (\varrho = n), \\[2ex] 0 & (\varrho > n). \end{cases}$$

P r o o f. Writing

$$(3.9) \qquad u_n = \frac{1}{(1*\alpha)_n}\sum_{\nu=0}^{n}\alpha_\nu s_\nu,$$

$$(3.9') \qquad t_n = \frac{1}{(1*\beta)_n}\sum_{\nu=0}^{n}\beta_\nu s_\nu,$$

we have $\Delta((1*\alpha)_n u_n) = \alpha_n s_n$ and so

$$t_n = \frac{1}{(1*\beta)_n}\sum_{\varrho=0}^{n}\frac{\beta_\varrho}{\alpha_\varrho}\Delta((1*\alpha)_\varrho u_\varrho) =$$

$$= \frac{1}{(1*\beta)_n}\left\{\sum_{\varrho=0}^{n-1}\left(\frac{\beta_\varrho}{\alpha_\varrho}-\frac{\beta_{\varrho+1}}{\alpha_{\varrho+1}}\right)(1*\alpha)_\varrho u_\varrho + \frac{\beta_n(1*\alpha)_n}{\alpha_n}u_n\right\} =$$

$$= \sum_{\varrho=0}^{n} d_{n,\varrho} u_\varrho,$$

where $d_{n,\varrho}$ is given by (3.8). If $s_n = 1$ for all n, then $u_n = 1$, $t_n = 1$, so that $\sum_{\varrho=0}^{n} d_{n,\varrho} = 1$ for every n.

Hence, it follows from Theorem 2 of [7], that $(\bar{N},\alpha) \subseteq (\bar{N},\beta)$ if and only if (3.6) and (3.7) hold.

4. Inclusion Theorems

T h e o r e m 1. If $(N,p,\alpha) \subseteq (N,q,\alpha)$ and

$$(4.1) \qquad (|k| * |(p*\Delta\alpha)|)_n = O((q*\Delta\alpha)_n),$$

then $[N,p,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda$ for $\lambda \geq 1$. When $\lambda = 1$, the condition (4.1) may be omitted.

P r o o f. Writing

$$L_n = \frac{(p * \Delta(\alpha s))_n}{(p * \Delta\alpha)_n} - s , \qquad M_n = \frac{(q * \Delta(\alpha s))_n}{(q * \Delta\alpha)_n} - s$$

and noting that $q_n = (k * p)_n$ (cf. (2.1)), we have

$$(q * \Delta\alpha)_n M_n = (k * (p * \Delta\alpha)L)_n .$$

Thus, using Hölder's inequality, and by (4.1), we obtain

$$(4.2) \quad \left\{|(q*\Delta\alpha)_n| \; |M_n|\right\}^\lambda \le \left\{(|k| * |(p*\Delta\alpha)| \, |L|)_n\right\}^\lambda \le$$

$$\le (|k| * |(p*\Delta\alpha)| \; |L|^\lambda)_n \left\{(|k| * |(p*\Delta\alpha)|)_n\right\}^{\lambda-1} \le$$

$$\le H(|k| * |(p*\Delta\alpha)| \, |L|^\lambda)_n \left\{|(q*\Delta\alpha)_n|\right\}^{\lambda-1} \le$$

$$\le H(|k| * |(p*\Delta\alpha)| \, |L|^\lambda)_n ,$$

and we have

$$\frac{1}{(q*\alpha)_n} \sum_{\nu=0}^{n} |(q*\Delta\alpha)_\nu| \; |M_\nu|^\lambda \le \frac{H}{(q*\alpha)_n} \sum_{\nu=0}^{n} |k_\nu| \sum_{\mu=0}^{n-\nu} |(p*\Delta\alpha)_\mu| \, |L_\mu|^\lambda .$$

We now suppose that $s_n \to s[N,p,\alpha]_\lambda$. Thus

$$X_n = \frac{1}{(p*\alpha)_n} \sum_{\nu=0}^{n} |(p*\Delta\alpha)_\nu| \; |L_\nu|^\lambda = o(1)$$

and hence

$$\frac{1}{(q*\alpha)_n} \sum_{\nu=0}^{n} |(q*\Delta\alpha)_\nu| \; |M_\nu|^\lambda \le \frac{H}{(q*\alpha)_n} \sum_{\nu=0}^{n} |k_{n-\nu}| \, (p*\alpha)_\nu \; X_\nu = o(1)$$

provided (3.4) and (3.5) hold. But, by Lemma 4, this is equivalent to the hypothesis $(N,p,\alpha) \subseteq (N,q,\alpha)$. Hence $s_n \to s[N,q,\alpha]_\lambda$ $s_n \to s[N,q,\alpha]_\lambda$ and the required inclusion follows.

Corollary. If $(N,p,\alpha) = (N,q,\alpha)$, then $[N,p,\alpha]_1 = [N,q,\alpha]_1$.

Theorem 2. If $\alpha_n > 0$, $q_n > 0$ for $n \geqslant 0$, $\Delta\alpha_n \geqslant 0$ for $n > 0$, $\{p_n\} \in \mu$, $p_n = O(q_n)$, (N,q,α) is regular and

$$\frac{q_n}{q_{n-1}} \leqslant \frac{p_n}{p_{n-1}} \qquad \text{for} \quad n > n_0,$$

then
$$[N,p,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda \qquad \text{for} \quad \lambda > 1.$$

Proof. Case $n_0 = 0$. Using Lemma 1 (ii), we find that

$$(|k| * p * \Delta\alpha)_n = 2k_0(p * \Delta\alpha)_n - (q * \Delta\alpha)_n = O((q * \Delta\alpha)_n)$$

since, whenever $\frac{q_n}{q_{n-1}} \leqslant \frac{p_n}{p_{n-1}}$ for $n > 0$, it can be easily verified that $p_n = O(q_n)$ implies $(p * \Delta\alpha)_n = O((q * \Delta\alpha)_n)$. Thus (4.1) holds and the result follows from Lemma 5 and Theorem 1.

For the general case, we have

$$\frac{q_n}{q_{n-1}} \leqslant \frac{p_n}{p_{n-1}} \qquad \text{for} \quad n = n_0+1, \ n_0+2, \ldots .$$

Write $\qquad t_n = q_n \qquad$ for $\quad n = n_0, n_0+1, n_0+2, \ldots,$

and define t_n recursively for $n = n_0-1, n_0-2, \ldots, 0$, so that $t_n > 0$ and

$$\frac{t_{n+1}}{t_n} \leqslant \min\left(\frac{t_{n+2}}{t_{n+1}},\ \frac{q_{n+1}}{q_n},\ \frac{p_{n+1}}{p_n}\right).$$

If we set $\xi_n = \frac{t_n}{t_0}$, then

$$\{\xi_n\}\epsilon\mu,\ \frac{\xi_n}{\xi_{n-1}} \leqslant \frac{q_n}{q_{n-1}},\ \frac{\xi_n}{\xi_{n-1}} \leqslant \frac{p_n}{p_{n-1}} \quad \text{for } n > 0,$$

and $q_n = O(\xi_n)$. Since $p_n = O(q_n) = O(\xi_n)$ and (N,ξ,α) is regular (by Lemma 2), therefore, by the case $n_0 = 0$ (with q_n replaced by ξ_n), we obtain

(4.3) $$[N,p,\alpha]_\lambda \subseteq [N,\xi,\alpha]_\lambda \quad \text{for } \lambda > 1.$$

Next, by Lemma 6 (with p_n replaced by ξ_n), we have $(N,\xi,\alpha) \subseteq (N,q,\alpha)$. If we write $\bar{k}(z) = \frac{q(z)}{\xi(z)}$, then, by Lemma 1(i), $\bar{k}_n \geqslant 0$ for $n \geqslant 0$; and so (4.1) holds. Thus, by Theorem 1,

(4.4) $$[N,\xi,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda \quad \text{for } \lambda > 1.$$

From (4.3) and (4.4), the required inclusion follows.

T h e o r e m 3. If, in addition to the hypotheses of Theorem 2, $\{q_n\}\epsilon\mu$, then $[N,p,\alpha]_\lambda = [N,q,\alpha]_\lambda$ for $\lambda > 1$.

P r o o f. Define $\{\xi_n\}$ as in the proof of Theorem 2. By Lemma 2, (N,ξ,α) is regular and so by the case $n_0 = 0$ of Theorem 2 (with p_n replaced by q_n and q_n replaced by ξ_n), we have $[N,q,\alpha]_\lambda \subseteq [N,\xi,\alpha]_\lambda$. Since, by Lemma 5, (N,p,α) is regular, therefore, from Lemma 6 and Theorem 1 (with p_n replaced by ξ_n and q_n replaced by p_n), we obtain $[N,\xi,\alpha]_\lambda \subseteq [N,p,\alpha]_\lambda$, because it can be easily verfied that, in this case also, (4.1) holds. Thus $[N,q,\alpha]_\lambda \subseteq [N,p,\alpha]_\lambda$ for $\lambda > 1$. This, in conjunction with Theorem 2, yields the result.

T h e o r e m 4. If $\alpha_n > 0$, $q_n > 0$ for $n \geqslant 0$, $\Delta\alpha_n \geqslant 0$ for $n > 0$, $\{p_n\}\epsilon\mu$, (N,p,α) and (N,q,α) are regular, and

$$\frac{p_n}{p_{n-1}} \leq \frac{q_n}{q_{n-1}} \quad \text{for} \quad n > n_o,$$

then

$$[N,p,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda \quad \text{for} \quad \lambda > 1.$$

For the case $n_o = 0$, the regularity of (N,p,α) is not needed.

P r o o f. When $n_o = 0$, by Lemma 1(i), $k_o^* > 0$, $k_n \geq 0$ for $n > 0$, and so (4.1) is satisfied. The result now follows from Lemma 6 and Theorem 1.

For the general case, interchanging p_n and q_n in the construction of $\{\xi_n\}$ (cf. the proof of Theorem 2), we, in this case, obtain

$$\{\xi_n\} \in \mu, \quad \frac{\xi_n}{\xi_{n-1}} \leq \frac{q_n}{q_{n-1}}, \quad \frac{\xi_n}{\xi_{n-1}} \leq \frac{p_n}{p_{n-1}}$$

for $n > 0$ and $p_n = O(\xi_n)$. Now, by Lemma 5, (N,ξ,α) is regular and so by the case $n_o = 0$ of Theorem 2 (with ξ_n in place of q_n), we have $[N,p,\alpha]_\lambda \subseteq [N,\xi,\alpha]_\lambda$; and by the case $n_o = 0$, $[N,\xi,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda$.

Thus $[N,p,\alpha]_\lambda \subseteq [N,q,\alpha]_\lambda$ for $\lambda > 1$, as desired.

5. Relations between generalised strong Nörlund, generalised absolute Nörlund and generalised Nörlund summability methods.

T h e o r e m 5. If (3.3) holds, then $[N,p,\alpha]_\lambda \subseteq (N,p,\alpha)$ for $\lambda \geq 1$.

P r o o f. Since, by Lemma 3 $[N,p,\alpha]_\lambda \subseteq [N,p,\alpha]_1$ for $\lambda > 1$, therefore it is enough to show that $[N,p,\alpha]_1 \subseteq (N,p,\alpha)$. Suppose $s_n \to s[N,p,\alpha]_1$. Thus

$$\frac{1}{(p*\alpha)_n} \sum_{\nu=0}^{n} |(p*\Delta\alpha)_\nu| \left| \frac{(p*\Delta(\alpha s))_\nu}{(p*\Delta\alpha)_\nu} - s \right| = o(1).$$

Now

$$\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}(p*\Delta\alpha)_\nu\left\{\frac{(p*\Delta(\alpha s))_\nu}{(p*\Delta\alpha)_\nu}-s\right\}=\frac{(p*\alpha s)_n}{(p*\alpha)_n}-s.$$

Thus

$$\left|\frac{(p*\alpha s)_n}{(p*\alpha)_n}-s\right|\leq\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}\left|(p*\Delta\alpha)_\nu\right|\left|\frac{(p*\Delta(\alpha s))_\nu}{(p*\Delta\alpha)_\nu}-s\right|=o(1)$$

and hence $s_n\to s(N,p,\alpha)$ and the result follows.

T h e o r e m 6. If $(\bar{N},p*\Delta\alpha)$ is regular and $\lambda\geq1$, then the necessary and sufficient conditions for a sequence $\{s_n\}$ to be summable $[N,p,\alpha]_\lambda$ to s are that it be summable (N,p,α) to s and that

$$\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}\left|(p*\Delta\alpha)_\nu\right|\left|L_\nu-\psi_\nu\right|^\lambda=o(1), \tag{5.1}$$

where

$$L_\nu=\frac{(p*\Delta(\alpha s))_\nu}{(p*\Delta\alpha)_\nu}, \tag{5.2}$$

$$\psi_\nu=\frac{(p*\alpha s)_\nu}{(p*\alpha)_\nu}. \tag{5.3}$$

P r o o f. Necessity. Suppose that $s_n\to s[N,p,\alpha]_\lambda$. Then, by Theorem 5, $s_n\to s(N,p,\alpha)$. Thus

$$\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}\left|(p,*\Delta\alpha)_\nu\right|\left|L_\nu-s\right|^\lambda=o(1) \tag{5.4}$$

and

$$\psi_n\to s\quad\text{as}\quad n\to\infty. \tag{5.5}$$

Since $(\bar{N}, p*\Delta\alpha)$ is regular, therefore, from (5.5), we obtain

$$(5.6) \qquad \frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|\psi_\nu - s|^\lambda = o(1).$$

By Minkowski's inequality, we obtain

$$(5.7) \qquad \left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|L_\nu - \psi_\nu|^\lambda\right\}^{\frac{1}{\lambda}} \leq$$

$$\leq \left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|L_\nu - s|^\lambda\right\}^{\frac{1}{\lambda}} +$$

$$+ \left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|\psi_\nu - s|^\lambda\right\}^{\frac{1}{\lambda}}.$$

Using (5.4) and (5.6) in (5.7), we obtain (5.1).

Sufficiency. Suppose that (5.1) and (5.5) hold. Then, in this case also, (5.6) holds. Hence, by Minkowski's inequality and (5.1),

$$\left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|L_\nu - s|^\lambda\right\}^{\frac{1}{\lambda}} \leq$$

$$\leq \left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|L_\nu - \psi_\nu|^\lambda\right\}^{\frac{1}{\lambda}} +$$

$$+ \left\{\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\;|\psi_\nu - s|^\lambda\right\}^{\frac{1}{\lambda}} = o(1)$$

so that (5.4) holds; and hence $s_n \to s\ [N, p, \alpha]_\lambda$.

Remarks

1) Since, in Theorem 6, $(\bar{N}, p*\Delta\alpha)$ is regular, therefore (5.1) is equivalent to

(5.8) $$\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\,|L_\nu-\psi_\nu|^\lambda = o(1).$$

2) If $\{s_n\}$ is summable $|N,p,\alpha|_p$, then $s_n \to s(N,p,\alpha)$, where

(5.9) $$s=\sum_{n=1}^{\infty}(t_n-t_{n-1})+t_0,$$

t_n being defined by (2.2).

T h e o r e m 7. If $(\bar{N},p*\Delta\alpha)$ is regular and $\{s_n\}$ is summable $|N, p,\alpha|_1$, then $s_n \to s[N,p,\alpha]_1$, where s is given by (5.9).

P r o o f. By Theorem 6 and Remark 1), it is enough to show that

$$\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\,|L_\nu-\psi_\nu| = o(1),$$

where L_ν, ψ_ν are defined by (5.2), (5.3), respectively. Now, for $n>0$,

$$L_n-\psi_n=\frac{(p*\alpha)_n(\Delta p*\alpha s)_n-(p*\Delta\alpha)_n(p*\alpha s)_n}{(p*\Delta\alpha)_n(p*\alpha)_n}=$$

$$=\frac{(p*\alpha)_n\left\{(p*\alpha s)_n-(p*\alpha s)_{n-1}\right\}-\left\{(p*\alpha)_n-(p*\alpha)_{n-1}\right\}(p*\alpha s)_n}{(p*\Delta\alpha)_n\,(p*\alpha)_n}=$$

$$=\frac{(p*\alpha)_{n-1}\,(p*\alpha s)_n-(p*\alpha)_n(p*\alpha s)_{n-1}}{(p*\Delta\alpha)_n\,(p*\alpha)_n}=$$

$$=\frac{(p*\alpha)_{n-1}\,\psi_n-(p*\alpha)_{n-1}\,\psi_{n-1}}{(p*\Delta\alpha)_n}.$$

Thus

$$(5.10)\qquad |(p*\Delta\alpha)_n|\,|L_n-\psi_n| = |(p*\alpha)_{n-1}|\,|\psi_n-\psi_{n-1}|\leqslant$$

$$\leqslant (1*|(p*\Delta\alpha)|)_{n-1}\,|\psi_n-\psi_{n-1}|$$

and hence

$$(5.11)\qquad \frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\,|L_\nu-\psi_\nu|\leqslant$$

$$\leqslant\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=1}^{n}(1*|(p*\Delta\alpha)|)_{\nu-1}\,|\psi_\nu-\psi_{\nu-1}|.$$

Writing $x_\nu=|\psi_\nu-\psi_{\nu-1}|$, $X_n=\sum_{\nu=1}^{n}x_\nu$

we find that the right hand side of (5.11) is

$$X_n-\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=1}^{n}|(p*\Delta\alpha)_\nu|\,X_\nu = o(1)$$

since $(\bar{N},p*\Delta\alpha)$ is regular. This completes the proof.

T h e o r e m 8. If $\lambda>1$, $(\bar{N},p*\Delta\alpha)$ is regular,

$$(5.12)\qquad (1*|(p*\Delta\alpha)|)_{n-1}=O(n|(p*\alpha)_n|)$$

and if $s_n\to s(N,p,\alpha)$ and $\{s_n\}$ is summable $|N,p,\alpha|_\lambda$, then

$$s_n\to s[N,p,\alpha]_\lambda\ .$$

P r o o f. By Theorem 6, it suffices to show that (5.8) holds. Now, using (5.10), we find, by (5.12), that

$$(5.13)\qquad \frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=0}^{n}|(p*\Delta\alpha)_\nu|\,|L_\nu-\psi_\nu|^\lambda\leq$$

$$\leq\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=1}^{n}\frac{(1*|(p*\Delta\alpha)|)_{\nu-1}^{\lambda}|\psi_\nu-\psi_{\nu-1}|^\lambda}{|(p*\Delta\alpha)_\nu|^{\lambda-1}}\leq$$

$$\leq\frac{1}{(1*|(p*\Delta\alpha)|)_n}\sum_{\nu=1}^{n}(1*|(p*\Delta\alpha)|)_{\nu-1}\,\nu^{\lambda-1}|\psi_\nu-\psi_{\nu-1}|^\lambda .$$

Now letting $y_\nu=\nu^{\lambda-1}|\psi_\nu-\psi_{\nu-1}|^\lambda$, $Y_n=\sum_{\nu=1}^{n}y_\nu$

and proceeding as in Theorem 7, we find that the right hand side of (5.13) is o(1). This establishes (5.8) and the proof is thus complete.

As particular instances of the results of Sections 4 and 5, when $\alpha_n=1$ for $n\geq 0$, we obtain various results contained in [4].

6. Relations between strong Riesz, absolute Riesz and Riesz summability methods

T h e o r e m 9. Let $\alpha_n\neq 0$ for $n\geq 0$. Then, for $\lambda\geq 1$,

$$(\bar{N},\alpha)\subseteq(\bar{M},\beta)\ \text{implies}\ [\bar{N},\alpha]_\lambda\subseteq[\bar{N},\beta]_\lambda .$$

P r o o f. Suppose that $s_n\to s[\bar{N},\alpha]_\lambda$. Thus, by definition,

$$(6.1)\qquad u_n=\frac{1}{(1*\alpha)_n}\sum_{\nu=0}^{n}|\alpha_\nu||s_\nu-s|^\lambda = o(1).$$

Writing

$$t_n=\frac{1}{(1*\beta)_n}\sum_{\nu=0}^{n}|\beta_\nu||s_\nu-s|^\lambda ,$$

and proceeding as in the proof of Lemma 7, we dotain

$$(6.2)\qquad t_n=\sum_{\varrho=0}^{n}d'_{n,\varrho}u_\varrho ,$$

where

$$d'_{n,\varrho} = \begin{cases} \left(\frac{|\beta_\varrho|}{|\alpha_\varrho|} - \frac{|\beta_{\varrho+1}|}{|\alpha_{\varrho+1}|}\right) \frac{(1*\alpha)_\varrho}{(1*\beta)_n} & (0 \leq \varrho \leq n-1), \\ \frac{|\beta_\varrho|}{|\alpha_\varrho|} \frac{(1*\alpha)_\varrho}{(1*\beta)_n} & (\varrho = n), \\ 0 & (\varrho > n). \end{cases}$$

Since $(\overline{N},\alpha) \subseteq (\overline{N},\beta)$, therefore (3.6) and (3.7) hold. Let

$$A_1 = \left\{\varrho:\ 0 \leq \varrho \leq n-1,\ \ \frac{|\beta_\varrho|}{|\alpha_\varrho|} \geq \frac{|\beta_{\varrho+1}|}{|\alpha_{\varrho+1}|}\right\},$$

$$A_2 = \left\{\varrho:\ 0 \leq \varrho \leq n-1,\ \ \frac{|\beta_\varrho|}{|\alpha_\varrho|} < \frac{|\beta_{\varrho+1}|}{|\alpha_{\varrho+1}|}\right\}.$$

Clearly $A_1 \cap A_2 = \varphi$ and $A_1 \cup A_2 = \{\varrho:\ 0 \leq \varrho \leq n-1\}$. Now

$$\sum_{\varrho=0}^{n} |d'_{n,\varrho}| = \sum_{\varrho \in A_1} \left(\left| \frac{|\beta_\varrho|}{|\alpha_\varrho|} - \frac{|\beta_{\varrho+1}|}{|\alpha_{\varrho+1}|} \right| \left| \frac{(1*\alpha)_\varrho}{(1*\beta)_n} \right| \right) +$$

$$+ \sum_{\varrho \in A_2} \left(\left| \frac{|\beta_\varrho|}{|\alpha_\varrho|} - \frac{|\beta_{\varrho+1}|}{|\alpha_{\varrho+1}|} \right| \left| \frac{(1*\alpha)_\varrho}{(1*\beta)_n} \right| + \frac{|\beta_n|}{|\alpha_n|} \left| \frac{(1*\alpha)_n}{(1*\beta)_n} \right| \right) \leq$$

$$\leq \sum_{\varrho \in A_1} \left(\left| \frac{|\beta_\varrho|}{|\alpha_\varrho|} - \frac{\beta_{\varrho+1}}{\alpha_{\varrho+1}} \right| \left| \frac{(1*\alpha)_\varrho}{(1*\beta)_n} \right| \right) +$$

$$+\sum_{\rho\in A_2}\left(\left|\frac{|\beta_{\rho+1}|}{|\alpha_{\rho+1}|}-\frac{|\beta_\rho|}{|\alpha_\rho|}\right|\left|\frac{(1*\alpha)_\rho}{(1*\beta)_n}\right|+\frac{|\beta_n|}{|\alpha_n|}+\frac{|(1*\alpha)_n|}{|(1*\beta)_n|}\right)\leqslant$$

$$\leqslant\sum_{\rho\in A_1}\left(\left|\frac{\beta_\rho}{\alpha_\rho}-\frac{\beta_{\rho+1}}{\alpha_{\rho+1}}\right|\left|\frac{(1*\alpha)_\rho}{(1*\beta)_n}\right|\right)+$$

$$+\sum_{\rho\in A_2}\left(\left|\frac{\beta_\rho}{\alpha_\rho}-\frac{\beta_{\rho+1}}{\alpha_{\rho+1}}\right|\left|\frac{(1*\alpha)_\rho}{(1*\beta)_n}\right|+\frac{|\beta_n|\,|(1*\alpha)_n|}{|\alpha_n|\,|(1*\beta)_n|}\right)=$$

$$=\sum_{\rho=0}^{n}|d_{n,\rho}|=O(1),$$

by (3.6). Also, for every fixed ρ, $d'_{n,\rho}\to 0$ as $n\to\infty$, by (3.7).

Thus the transformation (6.2) is null-preserving and hence, by (6.1), $t_n=o(1)$. Thus $s_n\to s\,[\bar{N},\beta]_\lambda$ and the required inclusion follows.

Putting $p_n=1$ for $n\geqslant 0$ in the results of Section 5, we obtain the following theorems.

T h e o r e m 10. If $\sum_{\nu=0}^{n}|\alpha_\nu|=O((1*\alpha)_n)$, then $[\bar{N},\alpha]_\lambda\subseteq(\bar{N},\alpha)$ for $\lambda\geqslant 1$.

T h e o r e m 11. Suppose that $(\bar{N},\alpha)$ is regular and $\lambda\geqslant 1$. Then $s_n\to s[\bar{N},\alpha]_\lambda$ if and only if $s_n\to s\ (\bar{N},\alpha)$ and

$$\frac{1}{(1*\alpha)_n}\sum_{\nu=0}^{n}|\alpha_\nu|\,|s_\nu-u_\nu|^\lambda=o(1),$$

where u_n is defined by (3.9).

T h e o r e m 12. If $(\bar{N},\alpha)$ is regular and $\{s_n\}$ is summable $|\bar{N},\alpha|_1$, then $s_n\to s[\bar{N},\alpha]_1$, where $s=\sum_{n=1}^{\infty}(u_n-u_{n-1})+u_0$, u_n being defined by (3.9).

T h e o r e m 13. Suppose that $\lambda > 1$, $(\bar{N},\alpha)$ is regular and $(1+|\alpha|)_{n-1} = O(n|(1+\alpha)_n|)$. If $s_n \to s(\bar{N},\alpha)$ and $\{s_n\}$ is summable $|\bar{N},\alpha|_\lambda$, then $s_n \to s[\bar{N},\alpha]_\lambda$.

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