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STABILITY OF THE DIFFERENCE SCHEMES FOR THE PARABOLIC TYPE EQUATION WITH THE MIXED DERIVATIVE

1. Introduction

This paper contains the proof of the stability of the difference schemes for the parabolic type partial differential equation with two variables.

Given the domain ω_{xy}, defined by inequalites $\underline{x} \leqslant x \leqslant \bar{x}$, $\underline{y} \leqslant y \leqslant \bar{y}$ and ω_t, by $0 \leqslant t \leqslant T$.
We denote the boundary of ω_{xy} by Γ and $\omega = \omega_{xy} \times \omega_t$.

Consider the equation

$$(1)\quad \frac{\partial u}{\partial t} = a_1(x,y,t)\frac{\partial^2 u}{\partial x^2} + 2\, a_2(x,y,t)\,\frac{\partial^2 u}{\partial x \partial y} + a_3(x,y,t)\frac{\partial^2 u}{\partial y^2} +$$

$$+ 2a_4(x,y,t)\frac{\partial u}{\partial x} + 2a_5(x,y,t)\,\frac{\partial u}{\partial y} + a_6(x,y,t)u + a_7(x,y,t),$$

for $(x,y,t) \in \omega$, and the conditions

$$(2)\qquad u(x,y,0) = f(x,y) \quad \text{for} \quad (x,y) \in \omega_{xy},$$

$$(3)\qquad u(x,y,t) = g(t) \quad \text{for} \quad (x,y\ t) \in \Gamma \times \omega_t,$$

$a_1, \ldots, a_7, f, g$ being given functions.

We choose three natural numbers n, m and s, and we define three real numbers (steps)

$$h_1 = \frac{\bar{x} - \underline{x}}{n}, \quad h_2 = \frac{\bar{y} - \underline{y}}{m}, \quad k = \frac{T}{s}.$$

The steps h_1, h_2 and k can not be choosen arbitrarily. The conditions, which must satisfy the numbers h_1, h_2, and k, shall be defined in the further considerations.

We denote

$$x_0 = \underline{x},\ x_i = x_0 + ih_1 \qquad \text{for } i = 0,1,\dots,n,$$

$$y_0 = \underline{y},\ y_j = y_0 + jh_2 \qquad \text{for } j = 0,1,\dots,m,$$

$$t_l = lk \qquad \text{for } l = 0,1,\dots,s,$$

$$u(x_i,y_j,t_l) = u_{ijl},$$

$$a_\nu(x_i,y_j,t_l) = a_{\nu ijl} \qquad \nu = 1,2,3,4,5,6,7,$$

$$h_1=\alpha_1 h,\ h_2 = \alpha_2 h,$$

where α_1, α_2 and h are real numbers. The points $P(x_i,y_j,t_l)$ are mesh-points.

2. Approximation

We obtain the following approximation of the derivatives in the mesh - points

$$\text{(4)}\quad \frac{\partial^2 u_{ijl}}{\partial x^2} = \frac{1}{\alpha_1^2 h^2}\Big[-q\ u_{i-1j-1l} + q\ u_{ij-1l}+(1+q)u_{i-1jl} + -2\ u_{ijl} + (1-q)\ u_{i+1jl} - q\ u_{ij+1l} + q\ u_{i+1j+1l}\Big]+ O_1(h),$$

$$\text{(5)}\quad \frac{\partial^2 u_{ijl}}{\partial x\,\partial y} = \frac{1}{\alpha_1\alpha_2\ h^2}\Big[(1-p)\ u_{i-1j-1l} - (1-p)\ u_{ij-1l} + - (1-p)\ u_{i-1jl}+u_{ijl}-p\ u_{i+1jl}-p\ u_{ij+1l}+p\ u_{i+1j+1l}\Big] +O_2(h),$$

$$(6)\quad \frac{\partial^2 u_{ijl}}{\partial y^2} = \frac{1}{\alpha_2^2 h^2}\Big[-r\, u_{i-1j-1l} + (1+r)\, u_{ij-1l} + r\, u_{i-1jl} + {}$$

$$- 2\, u_{ijl} - r\, u_{i+1jl} + (1-r)\, u_{ij+1l} + r\, u_{i+1j+1l}\Big] + O_3$$

$$(7)\quad \frac{\partial u_{ijl}}{\partial x} = \frac{1}{2\alpha_1 h}\Big[u_{i+1jl} - u_{i-1jl}\Big] + O_4(h)\ ,$$

$$(8)\quad \frac{\partial u_{ijl}}{\partial y} = \frac{1}{2\alpha_2 h}\Big[u_{ij+1l} - u_{ij-1l}\Big] + O_5(h),$$

$$(9)\quad \frac{\partial u_{ijl}}{\partial t} = \frac{1}{k}\Big[u_{ijl+1} - u_{ijl}\Big] + O_6(k).$$

The numbers p, q, r are parameters and can depend on the points P_{ijl}.

We denote

$$(10)\quad \begin{cases} b_{1ijl} = \dfrac{a_{1ijl}}{\alpha_1^2}, \\[2mm] b_{2ijl} = \dfrac{a_{2ijl}}{\alpha_1 \alpha_2}, \\[2mm] b_{3ijl} = \dfrac{a_{3ijl}}{\alpha_2^2}, \\[2mm] b_{4ijl} = \dfrac{a_{4ijl}}{\alpha_1}, \\[2mm] b_{5ijl} = \dfrac{a_{5ijl}}{\alpha_2}, \end{cases}$$

$$(10a)\quad \sigma = k : h^2.$$

Substituting the formulas (4) - (9) into the equation (1) and neglecting the rest-terms we obtain after transformation the finite difference equation

$$(11)\quad u_{ijl+1} = \left[1-\sigma(2b_{1ijl} - 2b_{2ijl}+2b_{3ijl}-a_{6ijl}h^2)\right]u_{ijl} +$$

$$+\sigma\left[-qb_{1ijl} - 2\ (1-p)b_{2ijl} - rb_{3ijl}\right] u_{i-1j-1l} +$$

$$+\sigma\left[qb_{1ijl} - 2(1-p)b_{2ijl} + (1+r)b_{3ijl} - b_{5ijl}h\right]u_{ij-1l} +$$

$$+\sigma\left[(1+q)b_{1ijl} - 2(1-p)b_{2ijl} + rb_{3ijl} - b_{4ijl}h\right]u_{i-1jl} +$$

$$+\sigma\left[(1-q)b_{1ijl} - 2pb_{2ijl} - rb_{3ijl} + b_{4ijl}h\right]u_{i+1jl} +$$

$$+\sigma\left[-qb_{1ijl} - 2pb_{2ijl} + (1-r)b_{3ijl} + b_{5ijl}h\right]u_{ij+1l} +$$

$$+\sigma\left[qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl}\right]u_{i+1j+1l} + \sigma a_{7ijl}h^2.$$

The error of this approximation of the equation (1) has the form

$$(12)\quad O_7(k+h) = a_{1ijl}O_1(h) + 2a_{2ij}O_2(h) + a_{3ijl}O_3(h) +$$

$$+ a_{4ijl}O_4(h^2) + a_{5ijl}O_5(h^2) - O_6(k).$$

The conditions for finite difference equation are

$$(13)\qquad u_{ij0} = f(x_i,y_j) \qquad \text{for} \quad (x_i,y_j)\in\omega_{xy},$$

$$(14)\qquad u_{ijl} = g(t_l) \qquad \text{for} \ (x_i,y_j,t_l)\in\Gamma\times\omega_t.$$

The points $P(x_0, y_j)$, $P(x_n, y_j)$ $(j=0,1,\dots,m)$ and $P(x_i, y_o)$, $P(x_i, y_m)$ $(i=0,1,\dots,n)$ belong to the boundary Γ.

3. The first condition of stability

Let the function u_{ijl} satisfy the difference equation (11) with the conditions (13) and (14) and let the functions V_{ijl} satisfy the equation (11) with the condition (14) and with

$$(15) \qquad V_{ij0} = f(x_i, y_j) + \varepsilon_{ij0} \quad \text{for} \quad (x_i, y_j) \varepsilon\, \omega_{xy}.$$

It is easy to see that the function

$$(16) \qquad W_{ijl} = V_{ijl} - u_{ijl}$$

satisfies the equation (11), with $a_{7ijl} = 0$ (i.e homogeneous equation) and fulfils the conditions

$$(17) \qquad W_{ij0} = \varepsilon_{ij0} \quad \text{for} \quad (x_i, y_j) \varepsilon \Gamma,$$

$$(18) \qquad W_{ijl} = 0 \quad \text{for} \quad (x_i, y_j, t_l) \epsilon \Gamma \times \omega_t .$$

Hence denoting

$$(19) \qquad \max_{i,j} |W_{ijl}| = \varepsilon^l .$$

we obtain

$$(20) \quad \varepsilon^{l+1} \leqslant \varepsilon^l \Big[\left|1 - \sigma(2b_{1ijl} - 2b_{2ijl} + 2b_{3ijl} - a_{6ijl}h^2)\right| +$$

$$+ \sigma \left| -qb_{1ijl} + 2(1-p)b_{2ijl} - rb_{3ijl} \right| +$$

$$+ \sigma \left| qb_{1ijl} - 2(1-p)b_{2ijl} + (1+r)b_{3ijl} - b_{5ijl}h \right| +$$

$$+ \sigma \left| (1+q)b_{1ijl} - 2(1-p)b_{2ijl} + rb_{3ijl} - b_{4ijl}h \right| +$$

$$+ \sigma \left|(1-q)b_{1ijl} - 2pb_{2ijl} - rb_{3ijl} + b_{4ijl}h\right| +$$

$$+ \sigma \left|-qb_{1ijl} - 2pb_{2ijl} + (1-r)b_{3ijl} + b_{5ijl}h\right| +$$

$$+ \sigma \left|qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl}\right|\Big].$$

We assume that

(21a) The coefficients $a_\nu(x,y,t)$ are continuous in ω.

(21b) $a_\nu(x,y,t) > 0$ for $\nu = 1,2,3$ and $(x,y,t) \in \omega$,

(21c) $a_2^2(x,y,t) - a_1(x,y,t) \leqslant 0$, $a_3(x,y,t)$ in ω,

(21d) The numbers α_1, α_2 and h are constant and independent of the point P_{ijl}.

Assuming furthermore

$$1 - \sigma(2b_{1ijl} - 2b_{2ijl} + 2b_{3ijl} - a_{6ijl}h^2) \geqslant 0, \tag{22}$$

$$-qb_{1ijl} + 2(1-p)b_{2ijl} - rb_{3ijl} \geqslant 0, \tag{23}$$

$$qb_{1ijl} - 2(1-p)b_{2ijl} + (1+r)b_{3ijl} - b_{5ijl}h \geqslant 0, \tag{24}$$

$$(1+q)b_{1ijl} - 2(1-p)b_{2ijl} + rb_{3ijl} - b_{4ijl}h \geqslant 0, \tag{25}$$

$$(1-q)b_{1ijl} - 2pb_{2ijl} - rb_{3ijl} + b_{4ijl}h \geqslant 0, \tag{26}$$

$$-qb_{1ijl} - 2pb_{2ijl} + (1-r)b_{3ijl} + b_{5ijl}h \geqslant 0, \tag{27}$$

$$qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl} \geqslant 0, \tag{28}$$

we get

$$\varepsilon^{l+1} \leqslant \varepsilon^{l}\,(1+\sigma a_{6ijl}h^2) \quad \text{for } l = 0,1,\ldots,s-1. \tag{29}$$

Denoting $\max\limits_{\omega} |a_6(x,y,t)| = \bar{a}_6$ and because $h^2 6 = k = \frac{T}{s}$ we have

$$\varepsilon^{l+1} \leq \varepsilon^{o} \left(1 + \frac{\bar{a}_6 T}{s}\right)^{l+1} \quad \text{for } l=0,1,\ldots,s-1.$$

Hence we obtain

$$\varepsilon^{l+1} \leq \varepsilon^{0} e^{\bar{a}_6 T} . \tag{30}$$

It is to see, that the function ε^{l+1} is limited and independent of s (i.e. from temporal step k). If the condition (30) is fulfiled, then the difference scheme is stable.

L e m m a 1. If the conditions (21b,c) are satisfied, then

$$b_{1ijl} - b_{2ijl} + b_{3ijl} > 0. \tag{31}$$

P r o o f. The assumption (21c) leeds to

$$a_{1ijl}\, a_{3ijl} > a^2_{2ijl} . \tag{32}$$

We divide the inequality (32) by $\alpha_1^2 \alpha_2^2$ and considering the expression (10) we obtain $b_{1ijl}\, b_{3ijl} > b^2_{2ijl}$.

Next we have

$$b^2_{1ijl} + 2b_{1ijl}b_{3ijl} + b^2_{3ijl} > b^2_{2ijl}+b^2_{1ijl}+b_{1ijl}b_{3ijl}+b^2_{3ijl} .$$

This implies (31).

We consider the expression

$$2b_{1ijl} - 2b_{2ijl} + 2b_{3ijl} - a_{6ijl}h^2. \tag{33}$$

If $a_{6ijl} < 0$, then the expression (33) is positive. If $a_{6ijl} > 0$, then the expression (33) is positive if and only if

$$h < \sqrt{\frac{2b_{1ijl} - 2b_{2ijl} + b_{3ijl}}{a_{6ijl}}} .$$

We denote

$$D(x,y,t) = \sqrt{\frac{\dfrac{2a_1(x,y,t)}{\alpha_1^2} - \dfrac{2a_2(x,y,t)}{\alpha_1\alpha_2} - \dfrac{2a_3(x,y,t)}{\alpha_2^2}}{a_6(x,y,t)}}$$

if $a_6\ x,y,t > 0$, and $D(x,y,t) = \infty$ if $a_6\ x,y,t \leq 0$,

$\bar{D} = \min D(x,y,t)$.

Hence we have the condition for step h. The inequality (22) leeds to

$$\sigma \leq \frac{1}{2b_{1ijl} - 2b_{2ijl} + 2b_{3ijl} - a_{6ijl}h^2} .$$

This implies the condition for σ

$$(35)\quad \sigma \leq \left\{ \max_{\omega} \left[\frac{2a_1(x,y,t)}{\alpha_1^2} - \frac{2a_2(x,y,t)}{\alpha_1\alpha_2} - \frac{2a_3(x,y,t)}{\alpha_2^2} - h^2 a_6(x,y,t) \right] \right\}^{-1} .$$

We have got two conditions of stability i.e. (34) and (35).

Now denoting

$$(36)\qquad S_{ijl} = qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl} ,$$

we can rewrite (23)-(28) in the form

$$(37)\qquad S_{ijl} \leq 2b_{2ijl} ,$$

$$(38)\qquad S_{ijl} \geq 2b_{2ijl} - b_{3ijl} + b_{5ijl}h ,$$

$$(39)\qquad S_{ijl} \geq 2b_{2ijl} - b_{1ijl} + b_{4ijl}h ,$$

$$(40)\qquad S_{ijl} \leq b_{1ijl} + b_{4ijl}h ,$$

$$(41)\qquad S_{ijl} \leq b_{3ijl} + b_{5ijl}h ,$$

(42) $$s_{ijl} \geq 0.$$

4. Stability in the case $a_4(x,y,t) = a_5(x,y,t) = 0$

In this case the inequalities (37)- (42) have the form

(43) $$s_{ijl} \leq 2b_{2ijl},$$

(44) $$s_{ijl} \geq 2b_{2ijl} - b_{3ijl},$$

(45) $$s_{ijl} \geq 2b_{2ijl} - b_{1ijl},$$

(46) $$s_{ijl} \leq b_{1ijl},$$

(47) $$s_{ijl} \leq b_{3ijl},$$

(48) $$s_{ijl} \geq 0.$$

We denote

(49) $$\delta_{ijl} = \min (2b_{2ijl}, b_{1ijl}, b_{3ijl}),$$

(50) $$\Delta_{ijl} = \max (0, 2b_{2ijl}-b_{1ijl}, 2b_{2ijl} - b_{3ijl}).$$

The conditions (43) - (48) hold if and only if

(51) $$\Delta_{ijl} \leq \delta_{ijl}.$$

L e m m a 2. If the conditions (21 a,b) are satisfied and if

(52) $$\max_{\omega} \frac{2a_2(x,y,t)}{a_3(x,y,t)} \leq \frac{\alpha_1}{\alpha_2} \leq \min_{\omega} \frac{a_1(x,y,t)}{2a_2(x,y,t)},$$

then

(53) $$2b_{2ijl} \leq b_{1ijl} \quad \text{and} \quad 2b_{2ijl} \leq b_{3ijl}.$$

P r o o f. From the inequality (52) we have

$$\frac{2a_{2ijl}}{a_{3ijl}} \leqslant \frac{\alpha_1}{\alpha_2} \qquad \text{and} \qquad \frac{\alpha_1}{\alpha_2} \leqslant \frac{a_{1ijl}}{2a_{2ijl}},$$

which leeds to (53).

T h e o r e m 1. If the conditions (21a,b,c,d),(34), (35) and (52) are satisfied, then there exist stable schémes for the equation (11).

P r o o f. Using (49), (50), and Lemma 2 we can write $\delta_{ijl} = 2b_{2ijl}$ and $\Delta_{ijl} = 0$. Therefore, by (43) - (48) we have

$$0 \leqslant S_{ijl} \leqslant 2b_{2ijl} \tag{54}$$

and hence, by (36) and (54), we obtain

$$0 \leqslant q\,\frac{b_{1ijl}}{2b_{2ijl}} + 2p + r\,\frac{b_{3ijl}}{2b_{2ijl}} \leqslant 1.$$

This holds, for example, if

$$q = \frac{b_{2ijl}}{2b_{1ijl}}, \quad p = \frac{1}{4}, \quad r = \frac{b_{2ijl}}{2b_{3ijl}}. \tag{55}$$

We can get the scheme with the constant coefficients, too. Namely denoting

$$\beta_1 = \max_{\omega} \frac{\alpha_2 a_1(x,y,t)}{2\alpha_1 a_2(x,y,t)}, \quad \gamma_1 = \min_{\omega} \frac{\alpha_2 a_1(x,y,t)}{2\alpha_1 a_2(x,y,t)},$$

$$\beta_2 = \max_{\omega} \frac{\alpha_1 a_3(x,y,t)}{2\alpha_2 a_2(x,y,t)}, \quad \gamma_2 = \min_{\omega} \frac{\alpha_1 a_3(x,y,t)}{2\alpha_2 a_2(x,y,t)},$$

we get

$$0 < \max_{ijl} \frac{b_{1ijl}}{2b_{2ijl}} \leqslant \beta_1 \quad \text{and} \quad 0 < \max_{ijl} \frac{b_{3ijl}}{2b_{2ijl}} \leqslant \beta_2,$$

$$0 < \gamma_1 \leq \min_{ijl} \frac{b_{1ijl}}{2b_{2ijl}}, \quad 0 < \gamma_2 \leq \min_{ijl} \frac{b_{3ijl}}{2b_{2ijl}} .$$

We can assume

(56) $$q + \frac{1}{4\beta_1}, \quad p = \frac{1}{4}, \quad r = \frac{1}{4\beta_2}.$$

Thus the inequalities (43) - (48) and (22) are satisfied. They present the sufficient conditions of stability of the difference scheme.

L e m m a 3. If the conditions (21 a,b) are satisfied and if

(57) $$\max_{\omega} \left(\frac{a_1(x,y,t)}{2a_2(x,y,t)}, \sqrt{\frac{a_1(x,y,t)}{a_3(x,y,t)}} \right) \leq \frac{\alpha_1}{\alpha_2} < \min_{\omega} \frac{a_1(x,y,t)}{a_2(x,y,t)},$$

then

(58) $$b_{2ijl} < b_{1ijl} \leq 2b_{2ijl} \quad \text{and} \quad b_{1ijl} \leq b_{3ijl}.$$

P r o o f. The inequality (57) leeds to

$$\frac{a_{1ijl}}{2a_{2ijl}} < \frac{\alpha_1}{\alpha_2}, \quad \frac{a_{1ijl}}{a_{3ijl}} \leq \frac{\alpha_1^2}{\alpha_2^2}, \quad \frac{\alpha_1}{\alpha_2} \leq \frac{a_{1ijl}}{a_{2ijl}},$$

and hence we have the conditions (58).

T h e o r e m 2. If the conditions (21a,b,c,d), (34), (35), and (57) are satisfied, then there exist the stable schemes for the equation (11).

P r o o f. According to (49) and (50) and to Lemma 3 we have

$$\delta_{ijl} = b_{1ijl} \quad \text{and} \quad \Delta_{ijl} = 2b_{2ijl} - b_{1ijl} .$$

Theorefore, by (43) - (48). the inequality $2b_{2ijl} - b_{1ijl} \leq \leq S_{ijl} \leq b_{1ijl}$ holds and, by (36),

(59) $$2b_{2ijl} - b_{1ijl} \leq qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl} \leq b_{1ijl} .$$

We can assume

$$qb_{1ijl} + 2pb_{2ijl} + rb_{3ijl} = b_{2ijl} ,$$

which implies

$$q\frac{b_{1ijl}}{2b_{2ijl}} + p + r\frac{b_{3ijl}}{2b_{2ijl}} = \frac{1}{2} .$$

This relation is satisfied, when, for example,

$$(60) \qquad q = \frac{b_{2ijl}}{4b_{1ijl}} , \quad p = \frac{1}{4} , \quad r = \frac{b_{2ijl}}{4b_{3ijl}} .$$

We can obtain the scheme with the constant coefficients. Namely, having (59) we get

$$1 - \frac{b_{1ijl}}{2b_{2ijl}} \leqslant q\frac{b_{1ijl}}{2b_{2ijl}} + p + r\frac{b_{3ijl}}{2b_{2ijl}} \leqslant \frac{b_{1ijl}}{2b_{2ijl}} .$$

We can denote

$$\frac{b_{1ijl}}{2b_{2ijl}} = \frac{1}{2} + A, \quad \text{where} \quad 0 < A \leqslant \frac{1}{2} , \quad \min A = \gamma_1 - \frac{1}{2}$$

which leeds to

$$- A \leqslant q\frac{b_{1ijl}}{2b_{2ijl}} + p - \frac{1}{2} + r\frac{b_{3ijl}}{2b_{2ijl}} \leqslant A.$$

Hence we get

$$(61) \qquad q = \frac{\gamma_1 - \frac{1}{2}}{4\beta_1}, \quad p = \frac{1}{2} , \quad r = \frac{\gamma_1 - \frac{1}{2}}{4\beta_2} .$$

In the same way we can prove the following theorem.

T h e o r e m 3. If the conditions (21a,b,c,d), (34), (35) are satisfied and if

$$\max_{\omega} \frac{a_2(x,y,t)}{a_3(x,y,t)} \leqslant \frac{\alpha_1}{\alpha_2} \leqslant \min_{\omega} \frac{a_1(x,y,t)}{a_3(x,y,t)}, \tag{62}$$

$$\frac{\alpha_1}{\alpha_2} < \min_{\omega} \frac{2a_2(x,y,t)}{a_3(x,y,t)}, \tag{63}$$

then there exist stable schemes for the equation (11).

The scheme with the variable coefficients has, for example the form

$$q = \frac{b_{2ijl}}{4b_{1ijl}}, \quad p = \frac{1}{4}, \quad r = \frac{b_{2ijl}}{4b_{3ijl}} \tag{64}$$

and with the constant ones

$$q = \frac{\gamma_2 - \frac{1}{2}}{4\beta_1}, \quad p = \frac{1}{2}, \quad r = \frac{\gamma_2 - \frac{1}{2}}{4\beta_2}. \tag{65}$$

5. <u>Stability in the case $a_4(x,y,t) \neq 0$ and $a_5(x,y,t) \neq 0$</u>

T h e o r e m 4. If the conditions (21a,b,c,d), (35) are satisfied and if

$$\max_{\omega} \frac{2a_2(x,y,t)}{a_3(x,y,t)} < \frac{\alpha_1}{\alpha_2} < \min_{\omega} \frac{a_1(x,y,t)}{2a_2(x,y,t)}, \tag{66}$$

$$h < \min_{\omega} \left(\bar{D}, \frac{a_1(x,y,t) - 2\frac{\alpha_1}{\alpha_2} a_2(x,y,t)}{\alpha_1 |a_4(x,y,t)|}, \frac{a_3(x,y,t) - 2\frac{\alpha_2}{\alpha_1} a_2(x,y,t)}{\alpha_2 |a_5(x,y,t)|} \right), \tag{67}$$

then there exist stable schemes for the equation (11).

P r o o f. In the same way as in Theorem 1 we get $2b_{2ijl} < b_{1ijl}$ and $2b_{2ijl} < b_{3ijl}$. From inequality (67) we have $2b_{2ijl} \leqslant b_{1ijl} - b_{4ijl}h$, $2b_{2ijl} \leqslant b_{3ijl} - b_{5ijl}h$.

Denoting

$$\delta_{ijl} = \min(2b_{2ijl}, b_{1ijl} + b_{4ijl}h, b_{3ijl} + b_{5ijl}h),$$

$$\Delta_{ijl} = \max(0, 2b_{2ijl} - b_{1ijl} + b_{4ijl}h, 2b_{2ijl} - b_{3ijl} + b_{5ijl}h),$$

we get $\delta_{ijl} = 2b_{2ijl}$ and $\Delta_{ijl} = 0$ and hence

$$0 \leqslant S_{ijl} \leqslant 2b_{2ijl}.$$

It is easy to see that the schemes defined by the equalities (55) and (56) can be applied in this case.

T h e o r e m 5. If the conditions (21a,b,c,d) and (35) are satisfied and if

$$\max_{\omega}\left(\frac{a_1(x,y,t)}{2a_2(x,y,t)}, \sqrt{\frac{a_1(x,y,t)}{a_3(x,y,t)}}\right) < \frac{\alpha_1}{\alpha_2} < \min_{\omega} \frac{a_1(x,y,t)}{a_2(x,y,t)}, \tag{68}$$

$$h < \min_{\omega}\left(\bar{D}, \frac{2\frac{\alpha_1}{\alpha_2} a_2(x,y,t) - a_1(x,y,t)}{\alpha_1 |a_4(x,y,t)|}, \right. \tag{69}$$

$$\left. \frac{a_1(x,y,t) - \frac{\alpha_1}{\alpha_2} a_2(x,y,t)}{\alpha_1 |a_4(x,y,t)|}, \frac{a_3(x,y,t)\frac{\alpha_1}{\alpha_2} - a_1(x,y,t)\frac{\alpha_2}{\alpha_1}}{\alpha_2|a_4(x,y,t)| + \alpha_1|a_5(x,y,t)|}\right),$$

then there exist stable schemes for the equation (11).

P r o o f. In the same way as in Theorem 2 the condition (68) implies $b_{2ijl} < b_{1ijl} < 2b_{2ijl}$ and $b_{1ijl} < b_{3ijl}$.

From (69) we obtain $2b_{2ijl} > b_{1ijl} \pm b_{4ijl}h$,

$b_{2ijl} > b_{1ijl} - b_{4ijl}h$, $b_{1ijl} + |b_{4ijl}|h < b_{3ijl} - |b_{5ijl}|h$

and hence we have

$$\delta_{ijl} = b_{1ijl} + b_{4ijl}h, \quad \Delta_{ijl} = 2b_{2ijl} - b_{1ijl} + b_{4ijl}h ,$$

$$2b_{2ijl} - b_{1ijl} + b_{4ijl}h \leq S_{ijl} \leq b_{1ijl} + b_{4ijl}h.$$

We can assume, $S_{ijl} = b_{2ijl}$.

It is easy to see that the schemes defined by the equalities (60) and (61) can be applied in this case.

In the same way as the Theorems 3 and 5 we can prove the following one.

T h e o r e m 6. If the conditions (21a,b,c,d) and (35) are satified and if

$$\max_{\omega}\left(\frac{a_2(x,y,t)}{a_3(x,y,t)}\right) < \frac{\alpha_1}{\alpha_2} < \min_{\omega}\left(\sqrt{\frac{a_1(x,y,t)}{a_3(x,y,t)}}\,,\ \frac{2a_2(x,y,t)}{a_3(x,y,t)}\right), \tag{70}$$

$$h < \min_{\omega}\left(\bar{D},\ \frac{2\frac{\alpha_2}{\alpha_1}a_2(x,y,t) - a_3(x,y,t)}{\alpha_2|a_5(x,y,t)|},\right. \tag{71}$$

$$\left.\frac{a_3(x,y,t) - \frac{\alpha_2}{\alpha_1}a_2(x,y,t)}{\alpha_2\,|a_5(x,y,t)|},\ \frac{a_1(x,y,t)\frac{\alpha_1}{\alpha_2} - a_3(x,y,t)\frac{\alpha_2}{\alpha_1}}{\alpha_2|a_4(x,y,t)| + \alpha_1\,|a_5(x,y,t)|}\right);$$

then there exist stable schemes for the equation (11). These schemes are given by the equalities (64) and (65).

Notice: If $a_2(x,y,t) = 0$, we can assume $q = p = 0$, (p is arbitrary). In this way we obtain the scheme which was considered in the paper [4].

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