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ON THE GENERALISED INVERTED DIRICHLET DISTRIBUTION

Generalised inverted Dirichlet distribution is discussed in relation to doubly non-central F variables. The distribution of the sum of the variables is obtained under certain conditions. Estimation and tests are suggested for these cases. Tail probabilities of the distribution of the sum are evaluated.

1. Introduction

Dirichlet's distribution and its generalised forms have received much attention recently as can be seen in [1], [2], [3], [4] and [5]. In most of these references, attention is mostly on the ratios $p_1, p_2, \ldots, p_k$, $0 < p_j < 1$, $j=1,\ldots,k$. For example [3], deals with the concept of "neutrality" of these quantities in relation to the independence of the terms like $p_j(1-p_1-p_2-\ldots - p_{j-1})^{-1}$, $j=1,2,\ldots,k$; [7] is concerned with the generalised Dirichlet distribution as applied to life testing via Bayesian approach along with variances and covariances; [4] specifically deals with the characterization of this distribution; [8] has used this distribution by introducing a concept of "F-independence"; [1] gives the characterization of the Gamma distribution in terms of p's. In this note, generalised inverted Dirichlet distribution is viewed through non-central F-variables. Further, as the quantities p_j $(1-p_1-\ldots -p_{j-1})^{-1}$ play an important role in most of the references cited, so also here quantities $x_j(1+x_1+\ldots+x_{j-1})^{-1}$, $x_j > 0$, $j=1,\ldots,k$, are dealt with throughout this paper. If the terms $p_j\ (1-p_1-\ldots-p_{j-1})^{-1}$ can be called as "force of mortality" of the population, as they are termed often, one could see the importance of the quantities used in this paper. Also, the distribution of the sum $x_1+\ldots+x_k$ is obtained under certain conditions and the estimation and tests are attempted in terms of this sum for these cases. The tail probabilities of the distribution of the sum are tabulated for the selected values of k and $a_1,a_2,\ldots,a_k$.

2. <u>The generalised inverted form</u>

2(i): Consider k independent doubly non-central F-variables

$$(1)\qquad F_j = f_j(\mu_j,\lambda_j;p_j,q_j) \qquad j=1,2,\ldots k,$$

where μ_j,λ_j are the parameters and p_j,q_j are the degrees of freedom. Let $p_j=q_j=2s_j$, then the joint density can be written as

$$(2)\qquad f(F_1,F_2,\ldots,F_k) = \sum_{l_1}\sum_{m_1}\ldots\sum_{l_k}\sum_{m_k}\prod_{j=1}^{k}\left[\exp\left(-\frac{1}{2}\sum_{j=1}^{k}(\lambda_j+\mu_j)\right)\times\left(\frac{\lambda_j}{2}\right)^{l_j}\left(\frac{\mu_j}{2}\right)^{m_j}\frac{1}{l_j!\ m_j!}\cdot\frac{F_j^{s_j+m_j-1}}{(1+F_j)^{2s_j+l_j+m_j}}\,\frac{1}{B(s_j+l_j;s_j+m_j)}\right],$$

$$0 < F_j < \infty\ ,\ j=1,\ldots k,$$

where 2k sums are on $l_1,m_1,\ldots l_k,m_k$ from 0 to ∞. Introducing the transformations

$$(3)\qquad F_j = \frac{x_j}{T_{j-1}} \text{ where } T_{j-1} = 1 + x_1+\ldots+x_{j-1},\ j=1,2,\ldots,k,$$

with $T_0=1$, we get in turn from (3)

(4) $$x_j = F_j \prod_{i=1}^{j-1} (1+F_i), \quad j=1,2,\ldots,k$$

and the Jacobian of transformation has the form

(5) $$|J| = \prod_{j=1}^{k} (1+F_j)^{k-j}.$$

Replacing (3) in (2) and taking $|J|$ along, we get (2) in the form

(6) $$f(x_1,\ldots,x_k) = \sum \prod_{j=1}^{k} \left[e^{-\frac{1}{2}\sum_{j=1}^{k}(\lambda_j+\mu_j)} \cdot \left(\frac{\lambda_j}{2}\right)^{l_j} \left(\frac{\mu_j}{2}\right)^{m_j} \frac{1}{l_j!\,m_j!} \frac{x_j^{s_j+m_j-1}}{B(s_j+l_j, s_j+m_j)\cdot T_j^{\theta_j}} \right]$$

where the sum $\sum$ is 2k-fold and

(7) $$\theta_j = 2s_j-(s_{j+1}+l_{j+1})+(l_j+m_j)$$

with $l_{k+1} = s_{k+1} = 0$. If $\lambda_j = \mu_j = 0$, $j = 1,2,\ldots,k$ (case of k-independent F-variables), we get (6) as

(8) $$f(x_1,\ldots,x_k) = \frac{x_1^{s_1-1} \cdots x_k^{s_k-1}}{\prod_{j=1}^{k}\left[B(s_j,s_j)\right] \cdot T_1^{2s_1-s_2} \cdots T_{k-1}^{2s_{k-1}-s_k} \cdot T_k^{2s_k}},$$

$$x_j > 0, \quad j=1,2,\ldots,k.$$

2(ii): General Case: (6) is actually a kind of an extension of the corresponding result in [7]. Because, in general, if we start with k-independent Beta variables $y_1,\ldots,y_k$, where

(8a) $$f(y_j) = \frac{y_j^{a_j-1}}{(1+y_j)^{a_j+\delta_j}}, \quad j=1,\ldots,k, \; y_j>0$$

and by a similar transformation (3) on y's we get

$$(9)\quad f(x_1,\dots,x_k) = \frac{x_1^{a_1-1}\dots x_k^{a_k-1}}{\prod_{j=1}^{k}\left[B(a_j,\delta_j)\right] T_1^{\varepsilon_1}\dots T_k^{\varepsilon_k}}, \quad x_j > 0,\ j=1,\dots,k,$$

where

$$(10)\qquad \varepsilon_j = \delta_j + a_j - \delta_{j+1}, \quad j = 1,\dots,k,$$

with $\delta_{k+1} = 0$. We can call (9) the generalised inverted Dirichlet distribution on the lines of [7] and [9]. If we set $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{k-1} = 0$ in (9), we get

$$(11)\quad f(x_1,x_2,\dots,x_k) = \frac{\left[x_1^{a_1-1}\dots x_k^{a_k-1}\right]\Gamma(\delta_1+a_1+\dots+a_k)}{(1+x_1+\dots+x_k)^{\delta_1+A}\,\Gamma(a_1)\dots\Gamma(a_k)\Gamma(\delta_1)},$$

$$x_j > 0, \quad j=1,2,\dots k,$$

where $A = a_1+\dots+a_k$, which, of course, is the inverted Dirichlet distribution.

2(iii): Now we find the characteristic function of the variables $\log x_1,\dots,\log x_k$, where $x_1,\dots,x_k$ appear in (9). We see in Section 4, that the variables $\log x_j$ play an important role. The characteristic function of these variables can be obtained either directly or by using the fact that $x_j = y_j \prod_{i=1}^{j-1}(1+y_i)$, where y_j's are independent Beta variables and we have

$$(12)\quad \phi(t_1,\dots,t_k) = \prod_{j=1}^{k}\left[\frac{\Gamma(a_j+it_j)\Gamma(\delta_j-\sum_{l=j}^{k} it_l)}{B(a_j,\delta_j)\Gamma(\delta_j+a_j-\sum_{l=j+1}^{k} it_l)}\right].$$

If we let $\varepsilon_1 = \dots = \varepsilon_{k-1}=0$ in (12), we get

$$(13) \qquad \phi(t_1,\ldots,t_k) = \prod_{j=1}^{k} \left[\frac{\Gamma(a_j+it_j)}{\Gamma(a_j)}\right] \cdot \frac{\Gamma\left(\delta_1 - \sum_{j=1}^{k} it_j\right)}{\Gamma(\delta_1)}$$

which obviously is the characteristic function of $\log x_1,\ldots,\log x_k$, where $x_1,\ldots,x_k$ are now those of (11). Actually x's in (11) are the ratios of two independent Gamma variables and (13) is the sufficiency condition for the characterization of the Gamma distribution which this author has discussed in detail in [6]. In all these ratios, denominator variable is the same. It may be noted here, that by starting with the Gamma distribution, both Dirichlet's distribution as well as its inverted form are obtained in [1] and [6] respectively. So also, starting with the Beta distribution, both the generalised Dirichlet distribution and its inverted form are derived in [3] and in this note respectively. The transformations used in [1] and [6] are different. But the transformations used in [3] and here look similar (in appearance only).

3. Distribution of the sum

3(a): Now we discuss the distribution of the sum $x_1+\ldots+x_k$. It is because of the facts, that by virtue of (3) we have

$$(14) \qquad T_k = 1 + x_1+\ldots+x_k = \prod_{j=1}^{k} (1+F_j)$$

and

$$(15) \qquad \sum_{j=1}^{k} \log x_j = \sum_{j=1}^{k} \log F_j + \sum_{j=1}^{k-1} (k-j)\log(1+F_j).$$

If F_j's are independent F-variables with equal degrees of freedom, say $2a_j$, then we have T's as the ratio of product of chi-squares each with $4a_j$ degrees of freedom to the product of chi-squares each with $2a_j$ degrees of freedom. It may also be noted here in this connection, that a similar sum $x_1+\ldots+x_k$ in (11) corresponds, under obvious conditions, to the ratio of a sum of independent chi-squares to a single chi-square which is of course again an F-variable. Now suppose that we set $\delta_j=a_j$ in (9), then we get (9) in the form

$$f(x_1,\dots,x_k) = \frac{C \cdot x_1^{a_1-1} \dots x_k^{a_k-1}}{T_1^{2a_1-a_2} \dots T_{k-1}^{2a_{k-1}-a_k} T_k^{2a_k}} \tag{16}$$

which naturally is (8), where $C = \left\{ \prod_{j=1}^{k} \left[B(a_j,a_j)\right]\right\}^{-1}$. Now, if we make the transformations

$$u_j = \sum_{i=1}^{j} x_i, \quad j = 1,\dots,k, \tag{17}$$

then (16) can be written as

$$f(u_1,\dots,u_k)=C\cdot \prod_{j=1}^{k} \left[\sum_{r_j=0}^{a_j-1} (-1)^{r_j}\binom{a_j-1}{r_j}\frac{1}{(1+u_j)^{\theta_j-\theta_{j+1}+1}}\right], \tag{18}$$

where $\theta_j = a_j + r_j$, with $a_{k+1} = r_{k+1} = \theta$, and $\theta_j \neq \theta_{j+1}$, $j = 1,\dots,k$.
Then

$$\int_0^{u_k}\int_0^{u_{k-1}} \dots \int_0^{u_2} f(u_1,\dots,u_k)du_1 \dots du_{k-1} \tag{19}$$

gives, upon setting $u_k=u$,

$$f(u)=C\cdot \prod_{j=1}^{k} \left[\sum_{r_j=0}^{a_j-1}(-1)^{r_j}\binom{a_j-1}{r_j}\sum_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k} \frac{1}{(\theta_j-\theta_i)}\ \frac{1}{(1+u)^{\theta_i+1}}\right], \tag{20}$$

$$u > 0,$$

plus the contribution, when $\theta_j = \theta_{j+1}$, $j = 1,\dots,k$ which is easily obtainable from (18).

Incidentally, in (20), we have

$$(21) \qquad \int_0^\infty f(u)\,du = C \cdot \prod_{j=1}^{k} \left[\sum_{r_j=0}^{a_j-1} (-1)^{r_j} \binom{a_j-1}{r_j} \frac{1 \qquad (|D|)}{|D|\,\theta_1\dots\theta_k} \right],$$

where $|D|$ is Vandermonde's determinant in θ's, that is,

$$|D| = \begin{vmatrix} 1 & \dots\dots & 1 \\ \theta_1 & \dots\dots & \theta_k \\ \vdots & & \vdots \\ \theta_1^{k-1} & \dots\dots & \theta_k^{k-1} \end{vmatrix}$$

and $\theta_j \neq \theta_{j+1}$ in (18) implies $\theta_i \neq \theta_j$ in $|D|$, and (21) is unity, since

$$(22) \qquad a \binom{2a-1}{a-1} \sum_{r=0}^{a-1} (-1)^r \binom{a-1}{r} \frac{1}{(a+r)} = 1 \,.$$

One could get a similar result as f(u) for the case, when δ's and a's are not equal in (9). But the case when δ's and a's are equal is more relevant because of F-variables than when they are not equal.

<u>3(b):</u> <u>$a_1 = a_2 = \dots = a_k = 1$.</u> If all a's are equal to 1 in (9), we get

$$(23) \qquad f(x_1,\dots,x_k) = \prod_{j=1}^{k} \delta_j \, T_j^{-(\delta_j - \delta_{j+1} + 1)}$$

with $\delta_{k+1} = 0$. The function (23) is exactly the same as (18) except that in the place of θ's we have now δ's. So, we get now

(24) $$f(u) = \left(\prod_{j=1}^{k} \delta_j\right) \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{1}{(\delta_j-\delta_i)} \frac{1}{(1+u)^{\delta_i+1}}, \quad u > 0,$$

with $\delta_i \neq \delta_j$ which is also necessary for (23). In this case again $\int_0^\infty f(u)du$ will involve Vandermonde's determinant in δ's. Actually the whole sum in (24) under the integral is $\frac{|D|}{\delta_1 \dots \delta_k |D|}$.

3(c): $\delta_1 = \dots = \delta_k = 1$. If all the δ's are equal to unity in (9), we get

(25) $$f(x_1,\dots x_k) = \frac{1}{T_k} \prod_{j=1}^{k} \left(a_j x_j^{a_j-1} T_j^{a_j}\right).$$

Now again with the transformation (17), we can write (25) as

(26) $$f(u_1,\dots,u_k) = \\ = \prod_{j=1}^{k} \left[a_j \sum_{r_j=0}^{a_j-1} (-1)^{a_j} \binom{a_j-1}{r_j} \frac{1}{(1+u_j)^{r_j-r_{j+1}+1}}\right] \frac{1}{(1+u_k)}$$

with $r_{k+1} = 0$. The function (26) is again of the same form as (18) with $r_j + 1$ in the place of Θ_j and now, using (19), we get

(27) $$f(u) = \prod_{j=1}^{k} \left[a_j \sum_{r_j=0}^{a_j-1} (-1)^{r_j} \binom{a_j-1}{r_j} \left\{\sum_{i=1}^{k} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{1}{(1+u)^{r_i+2}} \frac{1}{r_i-r_j}\right\}\right].$$

In this case $\int_0^\infty f(u)du$ is calculated by writing r_i-r_j as $(r_i+1)-(r_j+1)$ in (27) and by using the Vandermonde determinant in (r_j+1)'s. Namely, we obtain

$$(28) \qquad \prod_{j=1}^{k} \left[a_j \sum_{r_j=0}^{a_j-1} (-1)^{r_j} \binom{a_j-1}{r_j} \frac{1}{r_j+1} \right]$$

and (28) is equal to 1, since $a \sum_{r=0}^{a-1} \binom{a-1}{r} \frac{(-1)^r}{r+1} = 1$.

4. Estimation and tests

In this section we deal only with the cases (3b) and (3c) which are simple. The estimator and the test procedures are quite straightforward and elementary though the results may be little interesting.

4(a): Before estimation, we can put the density functions (23) and (25) in the standard useful forms. The function (23) can be put as

$$(29) \qquad f(x_1,\dots,x_k) = \exp\left[\sum_{j=1}^{k} \delta_j B_j(x) + D(\delta) + C(x) \right],$$

where

$$D(\delta) = \sum_{j=1}^{k} \log \delta_j, \quad C(x) = -\sum_{j=1}^{k} \log T_j \text{ and } B_j(x)-\log(T_j/T_{j-1})$$

with $T_0=1$ and, in the case of (25), we have

$$(30) \qquad f(x_1,\dots,x_k) = \exp\left[\sum_{j=1}^{k} a_j A_j(x)+E(x)+D(a) \right],$$

where

$$E(x) = -\log\left[(x_1 \dots x_k)T_k\right],$$

$$D(a) = \sum_{j=1}^{k} \log a_j \quad \text{and} \quad A_j(x) = \log \frac{x_j}{T_j}.$$

(4b): In the case of (3b) to test H_0: $\delta_1 = \dots = \delta_k = \delta$, we have the likelihood ratio criterion as

$$(31)\quad \frac{(k)^{nk} \prod_{j=1}^{k} \left[\sum_{i=1}^{n} \log\left(\frac{T_{ji}}{T_{j-1i}}\right)\right]^{n} \left[\prod_{j=1}^{k-1}\left(\prod_{i=1}^{n} T_{ji}^{\hat{\delta}_j-\hat{\delta}_{j+1}} T_{ki}^{\hat{\delta}_k-\hat{\delta}}\right)\right]}{\left(\sum_{i=1}^{n} \log T_{ki}\right)^{nk}},$$

where $\hat{\delta}_{k+1}=0$, $\hat{\delta} = \dfrac{nk}{\sum_{i=1}^{n} \log\ T_{ki}}$

and $\hat{\delta}_j = \dfrac{n}{\sum_{i=1}^{n} \log\left(\frac{T_{ji}}{T_{j-1,i}}\right)}$ $j=1,\ldots k.$

(4c): And in the case of (3c), to test the hypothesis H_0: $a_1 = a_2 = \ldots = a_k = a$, we have

$$(32)\quad \frac{(k)^{nk} \prod_{j=1}^{k} \left[\sum_{i=1}^{n} \log\left(\frac{T_{ji}}{x_{ji}}\right)\right]^{n} \left[\prod_{j=1}^{k} \prod_{i=1}^{n} T_{ji}^{\hat{a}_j-\hat{a}} \cdot X_{ji}^{\hat{a}-\hat{a}_j}\right]}{\left[\sum_{i=1}^{n} \sum_{j=1}^{k} \log\left(\frac{T_{ji}}{X_{ji}}\right)\right]^{nk}};$$

where

$$\hat{a} = nk\left[\sum_{i=1}^{n}\sum_{j=1}^{k} \log\left(\frac{T_{ji}}{x_{ji}}\right)\right]^{-1}$$

$$\hat{a}_j = n\left[\sum_{i=1}^{n} \log\left(\frac{T_{ji}}{x_{ji}}\right)\right]^{-1}.$$

The estimates of δ's in (31) are in agreement with those in (29). If we consider further special cases, when both δ's and a's are all equal to 1 (case, when all F's have degrees of freedom 2,2), we can replace $\hat{a}$, $\hat{a}_j$'s and $\hat{\delta}_j$'s in (31) and (32) in terms of F_j's, since then $\frac{T_j}{x_j} = \frac{1+F_j}{F_j}$.

There are few restrictions in our analysis. Firstly, only the integral values of a's are treated in (16) and (25) so as to deal with F-variables. Secondly, the estimation and tests are restricted to the cases when either all δ's are unity or all a's are unity. It would have been more useful if the same procedure of estimation had been applied to case (3a) also. But it is not so due to the term C and the same is in the general case (9) also.

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