DEMONSTRATIO MATHEMATICA
Vol. IX No 3 1976

Zofia Kareńska

ON SOME FUNCTIONAL EQUATION OCCURRING IN THE THEORY OF GEOMETRIC OBJECTS

0. Introduction

In this paper we find all solutions of the functional equation

$$g(x\cdot y) = F(x)\cdot g(y) + g(x), \tag{0.1}$$

where x and y are non-singular 2x2 real matrices i.e., $x,y \in GL(2,\mathbb{R})$, g is an unknown function whose values are 3x1 real matrices, and F is a given function of the form

$$F(x) = \begin{bmatrix} 1 & \alpha_1(\Delta_x) & \frac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x) \\ 0 & 1 & \alpha_1(\Delta_x) \\ 0 & 0 & 1 \end{bmatrix}, \tag{0,2^a}$$

or

$$F(x) = (\operatorname{sgn}\Delta_x)\begin{bmatrix} 1 & \alpha_1(\Delta_x) & \frac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x) \\ 0 & 1 & \alpha_1(\Delta_x) \\ 0 & 0 & 1 \end{bmatrix}, \tag{0.2^b}$$

where $\Delta_x = \det(x)$, α_1 and α_2 are arbitrary solutions of the functional equation

(0.3) $$\alpha(\xi\eta) = \alpha(\xi) + \alpha(\eta) \quad \text{for } \xi\eta \neq 0$$

with the additional restriction

(0.4) $$\alpha_1(\mathbb{R}\setminus\{0\}) \neq \{0\}.$$

We do not make any assumptions concerning the regularity of the function g.

Equation (0.1) appears in the theory of geometric objects when we want to find the solutions of the system of functional equations

$$F(x\cdot y) = F(x)\cdot F(y), \qquad g(x\cdot y) = F(x)\cdot g(y) + g(x)$$

([2], p.152) in order to determine the geometric objects of type [3,2,1] with linear non-homogeneous transformation rule.

The main result of the paper is Theorem 0.1 and Theorem 0.2.

T h e o r e m 0.1. The general solution of the functional equation (0.1) defined on GL(2,$\mathbb{R}$) in the case when the function F has the form $(0.2)^a$ is given by the formula

$(0.5)^a$ $$g(x) = \begin{bmatrix} \alpha_0(\Delta_x)+\tau\alpha_1^2(\Delta_x) + \frac{1}{3}\omega\alpha_1^3(\Delta_x)+2\omega\alpha_1(\Delta_x)\alpha_2(\Delta_x) \\ 2\tau\alpha_1(\Delta_x)+2\omega\alpha_2(\Delta_x)+\omega\alpha_1^2(\Delta_x) \\ 2\omega\alpha_1(\Delta_x) \end{bmatrix},$$

where ω, τ are real parameters and α_0 denotes an arbitrary function satisfying the equation (0.3).

T h e o r e m 0.2. The general solution of (0.1) defined on GL (2,$\mathbb{R}$) in the case when the function F has the form $(0.2)^b$ is given by the formula

$(0.5)^b$ $$g(x) = [F(x) - E]\cdot q,$$

where E is the unit 3x3 matrix i.e., $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

q is a 3x1 matrix whose entries are real parameters, F defined by $(0.2)^b$.

Furthermore from (0.3) we have for $i = 1,2$

$$\alpha_i(1) = \alpha_i(-1) = 0 \tag{0.6}$$

and

$$\alpha_i(\xi) = \alpha_i(-\xi) \quad \text{for every} \quad \xi \neq 0. \tag{0.7}$$

Let us observe that from the properties of the solutions of equation (0.3) in particular it follows that, if inequality (0.4) is fulfilled, then it is also valid if we confine ourselves to $\xi > 0$ only. Then

$$\alpha_1(\mathbb{R}_+) \neq \{0\}, \tag{0.8}$$

where $\mathbb{R}_+ = \{\xi \in \mathbb{R} : \xi > 0\}$.

1. The auxiliary lemmas

In the sequel of the present paper we shall apply the following lemmas

Lemma 1.1 (cf [5]) The general solution of the functional equation

$$\gamma(x \cdot y) = \varphi(\Delta_x)\gamma(y) + \gamma(x), \tag{1.1}$$

for all $x,y \in GL(2,\mathbb{R})$, where φ is an arbitrary not vanishing identically solution of equation

$$\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta) \quad \text{for all } \xi\eta \neq 0, \tag{1.1'}$$

is given by the formulae:

(1.2) $\gamma(x) = \lambda[\varphi(\Delta_x) - 1]$ if $\varphi \not\equiv 1$

and

(1.3) $\gamma(x) = \ln|\phi_0(\Delta_x)|$ if $\varphi \equiv 1$,

where $\Delta_x = \det(x)$, ϕ_0 is an arbitrary multiplicative function not vanishing identically i.e., ϕ_0 is an arbitrary function satisfying the equation $(1.1)'$ and the condition $\phi_0 \not\equiv 0$. is an arbitrary constant i.e., real parameter.

R e m a r k 1.1. From the properties of the solutions of equation $(1.1)'$ it follows that $\phi_0(\xi) \neq 0$ for every $\xi \neq 0$. Let us notice that $\ln|\phi_0|$ is an arbitrary function satisfying (0.3).

L e m m a 1.2 ([4] p.64, 65). The general solution of the system of equations

(1.4) $\omega_1(x\cdot y) = \omega_1(y) + \alpha(\Delta_x)\,\omega_2(y) + \omega_1(x)$

(1.5) $\omega_2(x\cdot y) = \omega_2(x) + \omega_2(y)$

for all $x,y \in GL(2,\mathbb{R})$, where α is an arbitrary not vanishing solution of the equation (0.3), is given by the formulae:

(1.6) $\omega_1(x) = \ln|\phi(\Delta_x)| + \omega\alpha^2(\Delta_x)$,

(1.7) $\omega_2(x) = 2\omega\alpha(\Delta_x)$,

where ω is arbitrary constant, ϕ is an arbitrary non-zero multiplicative function, $\Delta_x = \det(x)$.

L e m m a 1.3. [4] p.65, 66). The general solution of the system of equations

(1.8) $\omega_1(x.y) = (\operatorname{agn}\Delta_x)\omega_1(y) + (\operatorname{sgn}\Delta_x)\alpha(\Delta_x)\omega_2(y) + \omega_1(x)$

(1.9) $$\omega_2(x\cdot y) = (\operatorname{sgn}\Delta_x)\omega_2(y) + \omega_2(x)$$

for all $x,y \in GL(2,\mathbb{R})$, where α is an arbitrary not vanishing solution of the equation (0.3), is given by the formulae

(1.10) $$\omega_1(x) = \omega\,\alpha(\Delta_x)\operatorname{sgn}\Delta_x - \delta\left[\operatorname{sgn}\Delta_x - 1\right]$$

(1.11) $$\omega_2(x) = \omega\left[\operatorname{sgn}\Delta_x - 1\right],$$

where $\Delta_x = \det(x)$, ω , δ are arbitrary constants.

2. Proof of Theorem 0.1

A straightforward verification shows that the function defined by $(0.2)^a$ satisfies the equation $F(x\cdot y) = F(x)\cdot F(y)$ (cf. [3]) and any function of the form $(0.5)^a$ satisfies (0.1). Thus, it remains to prove that the function g satisfying equation 0,1 for all x,y GL $(2,\mathbb{R})$ must have the form $(0.5)^a$.

Let

(2.1) $$g(x) = \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \\ \gamma_3(x) \end{bmatrix}$$

be an arbitrary solution of the equation (0.1).

Let us notice that the matrix equation (0.1) is equivalent to the system of three equations:

(2.2) $$\gamma_1(x\cdot y) = \gamma_1(y)+\alpha_1(\Delta_x)\gamma_2(y)+\tfrac{1}{2}\left[\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right]\gamma_3(y) + \gamma_1(x)$$

(2.3) $$\gamma_2(x\cdot y) = \gamma_2(y) + \alpha_1(\Delta_x)\gamma_3(y) + \gamma_2(x)$$

(2.4) $$\gamma_3(x\cdot y) = \gamma_3(x) + \gamma_3(y).$$

It follows from (0.8) that there exists a real number $\beta > 0$ such that $\alpha_1(\beta) \neq 0$. The general solution of the system of the equations (2.3) and (2.4) has been given in Lemma 1.2 (cf. (1.6), (1.7)). Thus, in view Lemma 1.2 we have

(2.5) $$\gamma_2(x) = \ln|\phi_2(\Delta_x)| + \bar{\omega}\alpha_1^2(\Delta_x)$$

(2.6) $$\gamma_3(x) = 2\bar{\omega}\alpha_1(\Delta_x),$$

where

(2.7) $$2\bar{\omega} = \frac{\gamma_2(x_0)}{\alpha_1(\beta)}, \quad \Delta_x = \det(x), \qquad x_0 = \begin{bmatrix} \sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix},$$

ϕ_2 denotes here a multiplicative function not vanishing identically.

Taking into account the formulae (2.5) and (2.6) from (2.2) we obtain

(2.8) $$\gamma_1(x\cdot y) = \gamma_1(x) + \gamma_1(y) + \alpha_1(\Delta_x)\left[\ln|\phi_2(\Delta_y)| + \right.$$

$$\left. + \bar{\omega}\alpha_1^2(\Delta_y)\right] + \left[\frac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right] 2\bar{\omega}\alpha_1(\Delta_y).$$

Since $x\cdot x_0 = x_0\cdot x$ for every $x \in GL(2,\mathbb{R})$,

$$\gamma_1(x\cdot x_0) = \gamma_1(x_0\cdot x)$$

it follows from (2.8) that

$$\gamma_1(x) + \gamma_1(x_0) + \alpha_1(\Delta_x)\left[\ln|\phi_2(\beta)| + \bar{\omega}\alpha_1^2(\beta)\right] +$$

$$+\left[\frac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right] 2\bar{\omega}\alpha_1(\beta) = \gamma_1(x_0) + \gamma_1(x) +$$

$$+\alpha_1(\beta)\left[\ln|\phi_2(\Delta_x)| + \bar{\omega}\alpha_1^2(\Delta_x)\right] + \left[\frac{1}{2}\alpha_1^2(\beta) + \alpha_2(\beta)\right] 2\bar{\omega}\alpha_1(\Delta_x).$$

Thus,

$$2\bar{\omega}\alpha_1(\beta)\alpha_2(\Delta_x) + \alpha_1(\Delta_x)\ln|\phi_2(\beta)| - 2\bar{\omega}\alpha_1(\Delta_x)\alpha_2(\beta) =$$

$$= \alpha_1(\beta)\ln|\phi_2(\Delta_x)|.$$

Finally we obtain

$$\ln|\phi_2(\Delta_x)| = 2\bar{\tau}\alpha_1(\Delta_x) + 2\bar{\omega}\alpha_2(\Delta_x) \tag{2.9}$$

and from (2.5) we get

$$\gamma_2(x) = 2\bar{\tau}\alpha_1(\Delta_x) + 2\bar{\omega}\alpha_2(\Delta_x) + \bar{\omega}\alpha_1^2(\Delta_x), \tag{2.10}$$

where

$$2\bar{\tau} = \frac{\ln|\phi_2(\beta)| - 2\bar{\omega}\alpha_2(\beta)}{\alpha_1(\beta)}. \tag{2.11}$$

Thus finally from (2.9) or (2.10) we see that the equation (2.8) takes the form

$$\gamma_1(x\cdot y) = \gamma_1(x) + \gamma_1(y) + 2\bar{\tau}\alpha_1(\Delta_x)\alpha_1(\Delta_y) + \tag{2.12}$$

$$+ 2\bar{\omega}\alpha_1(\Delta_x)\alpha_2(\Delta_y) + \bar{\omega}\alpha_1(\Delta_x)\alpha_1^2(\Delta_y) +$$

$$+\bar{\omega}\alpha_1^2(\Delta_x)\alpha_1(\Delta_y) + 2\bar{\omega}\alpha_2(\Delta_x)\alpha_1(\Delta_y).$$

Now let us put

$$(2.13)\qquad \gamma_0(x) = \gamma_1(x) - \bar{\tau}\alpha_1^2(\Delta_x) - \frac{1}{3}\bar{\omega}\alpha_1^3(\Delta_x) - 2\bar{\omega}\alpha_1(\Delta_x)\alpha_2(\Delta_x)$$

The function (2.13) satisfies the equation

$$(2.14)\qquad \gamma_0(x\cdot y) = \gamma_0(x) + \gamma_0(y)$$

for all $x,y \in GL(2,\mathbb{R})$.

In fact, by (2.13), (2.12), (0.3) and in view of $\Delta_{x\cdot y} = \Delta_x \Delta_y$ we have

$$\gamma_0(x\cdot y) = \gamma_1(x\cdot y) - \bar{\tau}\alpha_1^2(\Delta_x\Delta_y) - \frac{1}{3}\bar{\omega}\left[\alpha_1(\Delta_x\Delta_y)\right]^3 -$$

$$- 2\bar{\omega}\alpha_1(\Delta_x\Delta_y)\alpha_2(\Delta_x\Delta_y) = \gamma_1(x\cdot y) - \bar{\tau}\left[\alpha_1(\Delta_x) + \alpha_1(\Delta_y)\right]^2 -$$

$$- \frac{1}{3}\left[\alpha_1(\Delta_x) + \alpha_1(\Delta_y)\right]^3 - 2\bar{\omega}\left[\alpha_1(\Delta_x) + \alpha_1(\Delta_y)\right]\left[\alpha_2(\Delta_x) + \alpha_2(\Delta_y)\right] =$$

$$= \left[\gamma_1(x) - \bar{\tau}\alpha_1^2(\Delta_x) - \frac{1}{3}\bar{\omega}\alpha_1^3(\Delta_x) - 2\bar{\omega}\alpha_1(\Delta_x)\alpha_2(\Delta_x)\right] +$$

$$+ \left[\gamma_1(y) - \bar{\tau}\alpha_1^2(\Delta_y) - \frac{1}{3}\bar{\omega}\alpha_1^3(\Delta_y) - 2\bar{\omega}\alpha_1(\Delta_y)\alpha_2(\Delta_y)\right] =$$

$$= \gamma_0(x) + \gamma_0(y).$$

Since for $x,y \in GL(2,\mathbb{R})$ the function γ_0 satisfies (2.14) then from Lemma 1.1, when $\varphi \equiv 1$, it follows that $\gamma_0(x) = \ln|\phi_0(\Delta_x)|$. But $\ln|\phi_0|$ is some solution of the equation (0.3) thus

$$(2.15)\qquad \gamma_0(x) = \alpha_0(\Delta_x)$$

where α_0 denotes a solution (0.3). Applying the formula (2.15) to (2.13) we obtain

$$\gamma_1(x) = \alpha_0(\Delta_x) + \bar{\tau}\alpha_1^2(\Delta_x) + \frac{1}{3}\bar{\omega}\alpha_1^3(\Delta_x) + \tag{2.16}$$

$$+ 2\bar{\omega}\alpha_1(\Delta_x)\alpha_2(\Delta_x).$$

Thus, summing up the results of the above considerations (cf.(2.16), (2.10), (2.6))we conclude that g satisfying equation (0.1) must have the form $(0.5)^a$.

3. Proof of Theorem 0. 2.

A straightforward verification shows that the function defined by $(0.2)^b$ satisfies the equation $F(x\cdot y) = F(x)\cdot F(y)$ (cf.[3]) and any function g of the form $(0.5)^b$ satisfies (0.1). In fact, $g(x\cdot y) = [F(x.y) - E]\cdot q = F(x)\cdot F(y)\cdot q - F(x)\cdot q + F(x)\cdot q - E\cdot q = F(x)\cdot[F(y) - E]\cdot q + [F(x)-E]\cdot q = F(x)\cdot g(y) + g(x)$.

Thus, it remains to prove that the function g satisfying equation (0.1) for all $x,y \in GL(2,\mathbb{R})$ must have the form $(0.5)^b$.

Let

$$g(x) = \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \\ \gamma_3(x) \end{bmatrix} \tag{3.1}$$

be an arbitrary solution of the equation (0.1).

Let us notice that the equation (0.1) is equivalent to the system of three equations

$$\gamma_1(x\cdot y) = (\mathrm{sgn}\Delta_x)\gamma_1(y) + \alpha_1(\Delta_x)\,(\mathrm{sgn}\Delta_x)\gamma_2(y) + \tag{3.2}$$

$$+ (\mathrm{sgn}\Delta_x)\left[\frac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right]\gamma_3(y) + \gamma_1(x)$$

(3.3) $$\gamma_2(x\cdot y) = (\operatorname{sgn}\Delta_x)\gamma_2(y) + (\operatorname{sgn}\Delta_x)\alpha_1(\Delta_x)\gamma_3(y)+\gamma_2(x)$$

(3.4) $$\gamma_3(x\cdot y) = (\operatorname{sgn}\Delta_x)\gamma_3(y) + \gamma_3(x).$$

Then according to Lemma 1.3 the solution of the functional system of equations (3.3) and (3.4) is given by the following formulae

(3.5) $$\gamma_2(x) = \gamma\alpha_1(\Delta_x)\ \operatorname{sgn}\Delta_x - \delta\left[\operatorname{sgn}\Delta_x - 1\right]$$

and

(3.6) $$\gamma_3(x) = \gamma\left[\operatorname{sgn}\Delta_x - 1\right],$$

where γ, δ are constants.

Taking into account (3.5) and (3.6) in (3.2) we obtain

$$\gamma_1(x\cdot y) = (\operatorname{sgn}\Delta_x)\gamma_1(y) + \alpha_1(\Delta_x)(\operatorname{sgn}\Delta_x)\left[\gamma\alpha_1(\Delta_y)\operatorname{sgn}\Delta_y - \delta(\operatorname{sgn}\Delta_y - 1)\right] + (\operatorname{sgn}\Delta_x)\left[\tfrac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right]\gamma\ (\operatorname{sgn}\Delta_y - 1) + \gamma_1(x).$$

Hence

(3.7) $$\begin{aligned}\gamma_1(x\cdot y) &= (\operatorname{sgn}\Delta_x)\gamma_1(y) + \gamma_1(x) + \\ &+ \gamma\alpha_1(\Delta_x)\alpha_1(\Delta_y)(\operatorname{sgn}\Delta_x)\ \operatorname{sgn}\Delta_y - \\ &- \delta\alpha_1(\Delta_x)(\operatorname{sgn}\Delta_y - 1)\ \operatorname{sgn}\Delta_x + \gamma\left[\tfrac{1}{2}\alpha_1^2(\Delta_x) + \right. \\ &\left. + \alpha_2(\Delta_x)\right](\operatorname{sgn}\Delta_y - 1)\ \operatorname{sgn}\Delta_x.\end{aligned}$$

Now let us put

$$(3.8)\qquad \gamma_0(x) = \gamma_1(x) - \tfrac{1}{2}\gamma\alpha_1^2(\Delta_x)\ \mathrm{sgn}\Delta_x + \delta\alpha_1(\Delta_x)\mathrm{sgn}\,\Delta_x - $$
$$- \gamma\alpha_2(\Delta_x)\ \mathrm{sgn}\,\Delta_x\ .$$ [1]

Let us notice, that the function γ_0 satisfy the equation

$$(3.9)\qquad \gamma_0(x\cdot y) = \gamma_0(y)\ \mathrm{sgn}\,\Delta_x + \gamma_0(x).$$

In fact, we have from (3.7) and (3.8)

$$\gamma_0(x\cdot y) = \gamma_1(x\cdot y) - \tfrac{1}{2}\gamma\alpha_1^2(\Delta_{x\cdot y})\mathrm{sgn}\Delta_{x\cdot y} + \delta\alpha_1(\Delta_{x\cdot y})\mathrm{sgn}\,\Delta_{x\cdot y} -$$
$$-\gamma\alpha_2(\Delta_{x\cdot y})\mathrm{sgn}\,\Delta_{x\cdot y} = \gamma_1(y)\ \mathrm{sgn}\,\Delta_x + \gamma_1(x) +$$
$$+\gamma\alpha_1(\Delta_x)\alpha_1(\Delta_y)(\mathrm{sgn}\,\Delta_x)\mathrm{sgn}\,\Delta_y - \delta\alpha_1(\Delta_x)(\mathrm{sgn}\,\Delta_y - 1)\ \mathrm{sgn}\,\Delta_x +$$
$$+\gamma\left[\tfrac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right](\mathrm{sgn}\,\Delta_y - 1)\ \mathrm{sgn}\,\Delta_x -$$
$$-\tfrac{1}{2}\gamma\left[\alpha_1^2(\Delta_x) + 2\,\alpha_1(\Delta_x)\,\alpha_1(\Delta_y) + \alpha_1^2(\Delta_y)\right](\mathrm{sgn}\,\Delta_x)\ \mathrm{sgn}\,\Delta_y +$$
$$+\delta\left[\alpha_1(\Delta_x) + \alpha_1(\Delta_y)\right](\mathrm{sgn}\,\Delta_x)\ \mathrm{sgn}\,\Delta_y -$$
$$-\gamma\left[\alpha_2(\Delta_x) + \alpha_2(\Delta_y)\right]\ (\mathrm{sgn}\,\Delta_x)\ \mathrm{sgn}\,\Delta_y =$$

---

[1] The function γ_0 (cf.(3.8) has been suggested by M.Kuczma.

$$= (\operatorname{sgn}\Delta_x)\Big[\gamma_1(y) - \tfrac{1}{2}\alpha_1^2(\Delta_y)\operatorname{sgn}\Delta_y + \delta\alpha_1(\Delta_y)\operatorname{sgn}\Delta_y -$$

$$- \gamma\alpha_2(\Delta_y)\operatorname{sgn}\Delta_y\Big] + \Big[\gamma_1(x) - \tfrac{1}{2}\alpha_1^2(\Delta_x)\operatorname{sgn}\Delta_x +$$

$$+ \delta\alpha_1(\Delta_x)\operatorname{sgn}\Delta_x - \gamma\alpha_2(\Delta_x)\operatorname{sgn}\Delta_x\Big] = \gamma_0(y)\operatorname{sgn}\Delta_x + \gamma_0(x).$$

From (3.9) taking into account Lemma 1.1 [cf. (1.2), when $\varphi(\Delta_x) = \operatorname{sgn}\Delta_x$] we obtain

$$\gamma_0(x) = \Theta\left[\operatorname{sgn}\Delta_x - 1\right], \tag{3.10}$$

where Θ is constant.

Applying (3.10) to the relation (3.8) we have

$$\gamma_1(x) = \Theta\left[\operatorname{sgn}\Delta_x - 1\right] + \tfrac{1}{2}\gamma\alpha_1^2(\Delta_x)\operatorname{sgn}\Delta_x - \delta\alpha_1(\Delta_x)\operatorname{sgn}\Delta_x +$$

$$+ \gamma\alpha_2(\Delta_x)\operatorname{sgn}\Delta_x .$$

Finally we obtain

$$\gamma_1(x) = \gamma\left[\tfrac{1}{2}\alpha_1^2(\Delta_x) + \alpha_2(\Delta_x)\right]\operatorname{sgn}\Delta_x - \delta\alpha_1(\Delta_x)\operatorname{sgn}\Delta_x + \Theta\left[\operatorname{sgn}\Delta_x - 1\right], \tag{3.11}$$

where γ, δ, Θ are constants.

Summing up the above results (3.5), (3.6) and (3.11) we have

$$g(x) = \begin{bmatrix} \gamma\left[\tfrac{1}{2}\alpha_1^2(\Delta_x)+\alpha_2(\Delta_x)\right]\operatorname{sgn}\Delta_x - \delta\alpha_1(\Delta_x)\operatorname{sgn}\Delta_x + \Theta\left[\operatorname{sgn}\Delta_x - 1\right] \\ \gamma\alpha_1(\Delta_x)\operatorname{sgn}\Delta_x - \delta\left[\operatorname{sgn}\Delta_x - 1\right] \\ \gamma\left[\operatorname{sgn}\Delta_x - 1\right] \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{sgn}\Delta_x - 1 & \alpha_1(\Delta_x)\operatorname{sgn}\Delta_x & \left[\frac{1}{2}\alpha_1^2(\Delta_x)+\alpha_2(\Delta_x)\right]\operatorname{sgn}\Delta_x \\ 0 & \operatorname{sgn}\Delta_x - 1 & \alpha_1(\Delta_x)\operatorname{sgn}\Delta_x \\ 0 & 0 & \operatorname{sgn}\Delta_x - 1 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ -\delta \\ \gamma \end{bmatrix} =$$

$$= \left[F(x) - E\right] \cdot q \,, \quad \text{where} \quad q = \begin{bmatrix} \theta \\ -\delta \\ \gamma \end{bmatrix}.$$

Thus, $g(x)$ is the form $(0.5)^b$ and the proof of Theorem 0.2 has been completed.

REFERENCES

[1] J. A c z é l: Vorlesungen über Funktionalgleichungen und ihre Anwendungen, Basel-Stuttgart 1961.

[2] J. A c z é l, S. G o ł ą b: Funktionalgleichungen der Theorie der geometrischen Objekte, Warszawa 1960.

[3] Z. K a r e ń s k a: O równaniu funkcyjnym $F(x.y) = F(x)\cdot F(y)$, Zeszyty Nauk.Politech. Krakow. 6 (1972) 3-269.

[4] M. K u c h a r z e w s k i, M. K u c z m a: On a system of functional equations occurring in the theory of geometric objects, Ann.Polon.Math. 14 (1963) 59 - 67.

[5] M. K u c z m a: On linear differential geometric objects of the first class with one component, Publ.Math.Debrecen 6 (1959) 72 - 78.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, KRAKÓW

Received June 16, 1975.