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SEMI-UNIFORM CONVERGENCE OF A SEQUENCE OF FUNCTIONS

1. Introduction

A well-known theorem from analysis asserts that the limit of uniformly convergent sequence of continuous functions is a continuous function. By a detailed analysis of the standard proof of this theorem we see that it remains true, when we everywhere replace word "continuous" by "upper semi-continuous". In the modified theorem we may replace the assumption that the sequence is uniformly convergent by the upper (lower) uniform convergence which is defined as follows.

D e f i n i t i o n 1. A sequence of functions $\{f_n\}$ convergent to f is said to be upper uniform convergent in the set X, if

$$\bigwedge_{\varepsilon>0} \bigvee_{N_\varepsilon} \bigwedge_{x\in X} \bigwedge_{n>N_\varepsilon} f_n(x) - f(x) < \varepsilon .$$

Lower uniform convergence is defined analogously. An upper (lower) uniform convergent sequence is said to be semi-uniform convergent.

The uniform convergence can be characterized by some properties of functions which, if possessed by terms, are possessed by the limit function, too (cf. [1]). In the first part of the present paper we give a similar characterization of the upper uniform convergence (the lower uniform convergence can be characterized analogously). The second part contains an analysis of a certain result of E.P. Rozycki [2] which generalizes Egoroff's theorem. He deduced the uniform convergence on a certain set from some assumptions, one of which states, in fact, the upper (lower) uniform convergence. In the present paper we show that the other assumptions of the theorem of Rozycki imply the lower (upper) uniform convergence.

2. Characterization of semi-uniform convergence

T h e o r e m 1. Let E be an arbitrary countably additive family of subsets of a set X. If a sequence $\{f_n\}$ is upper uniformly convergent on X to a function f and if for every $n \in N$ and $a \in R$ we have

$$\{x : f_n(x) > a\} \in E, \tag{1}$$

then for every a we have

$$\{x : f(x) > a\} \in E. \tag{2}$$

P r o o f. Let $\{j_n\}$ be a fixed sequence of positive integers such that for every $x \in X$ we have

$$f_{j_n+k}(x) - f(x) < \frac{1}{n} \quad \text{for every} \quad k \in N. \tag{3}$$

The existence of the sequence $\{j_n\}$ follows from the upper uniform convergence of the sequence $\{f_n\}$. We observe that for every a the set

$$\{x : f(x) > a\} = \bigcup_n \bigcup_k \left\{x : f_{j_n+k}(x) > a + \frac{1}{n}\right\}$$

belongs to E as a countable sum of sets belonging to E, which completes the proof.

An analogous theorem holds for lower uniform convergence. Restricting the assumption of Theorem 1 to numbers a in some interval and to some subsequence we can obtain a somewhat more general theorem.

T h e o r e m 1'. If a sequence $\{f_n\}$ is upper uniformly convergent on a certain set X to a function f, then for every interval (y_1,y_2), every countably additive family E of subsets of X and every increasing sequence of positive integers $\{n_k\}$ the condition

(1') $$\left\{x : f_{n_k}(x) > a\right\} \in E \quad \text{for } a \in (y_1,y_2)$$

implies the condition

(2') $$\left\{x : f(x) > a\right\} \in E \text{ for } a \in (y_1,y_2).$$

The converse to Theorem 1' holds under the additional assumption that all functions f_n $(n \in N)$ are uniformly bounded.

T h e o r e m 2. If functions f_n $(n \in N)$ are uniformly bounded and the sequence $\{f_n\}$ converges on X to a function f and this convergence is not upper uniform, then there exist

a) an interval (y_1,y_2),

b) a countably additive family E of subsets of X,

c) an increasing sequence $\{n_k\}$ of positive integers such that $\left\{x : f_{n_k}(x) > a\right\} \in E$ for $a \in (y_1,y_2)$, $k \in N$ and $\left\{x : f(x) > a\right\} \notin E$ for $a \in (y_1,y_2)$.

P r o o f. Since the sequence $\{f_n\}$ is not upper uniformly convergent to f, there exist a positive number ε_0, an increasing sequence $\{n_k\}$ of positive integers and a sequence $\{x_k\}$, $x_k \in X$ for $k \in N$ such that

(4) $$f_{n_k}(x_k) - f(x_k) > \varepsilon_0 \quad \text{for} \quad k \in N.$$

Choosing, if necessary, convenient subsequences, we may suppose that the sequences $\left\{f_{n_k}(x_k)\right\}$, $\left\{f(x_k)\right\}$ are convergent. We put

$$y_1 = \lim_{k\to\infty} f(x_k) + \frac{\varepsilon_0}{3}, \quad y_2 = \lim_{k\to\infty} f_{n_k}(x_k) - \frac{\varepsilon_0}{3}.$$

Omitting, if necessary, a finite number of terms we may assume that

$$(5) \qquad f(x_k) < y_1, \qquad \text{for } k = 1,2,\ldots$$

and

$$(6) \qquad f_{n_k}(x_k) > y_2. \qquad \text{for } k - 1,2,\ldots$$

Let E be the family of those subsets of X which contain at least one term x_k (E is evidently a countably additive family). In virtue of inequality (6) we have

$$\{x : f_{n_k}(x) > a\} \in E \quad \text{for } a \in (y_1, y_2)$$

but from inequality (5) it follows that

$$\{x : f(x) > a\} \notin E \quad \text{for } a \in (y_1, y_2).$$

Theorems 1′ and 2 allow us to characterize the upper uniform convergence. Before we do this, we introduce the following definition.

D e f i n i t i o n 2. A property of function f is said to be of the type of lower measurability of there exist a countably additive family E of subsets of X and an interval (y_1, y_2) such that f has this property if and only if for every $a \in (y_1, y_2)$ we have $\{x : f(x) > a\} \in E$.

T h e o r e m 3. Let functions f_n $(n \in N)$ be uniformly bounded and let the sequence $\{f_n\}$ converge to a function f. This convergence is upper uniform if and only if the following implication is true:
if infinitely many terms of sequence $\{f_n\}$ have a property of the type of lower measurability, then the limit function has the same property.

One can characterize analogously the lower uniform convergence.

3. Remarks on Rozycki's theorem

According to E.P. Rozycki we use the following notations. Let m be a measure defined on a certain countably additive family of subsets of X. Let A be a set of finite measure and let F be a real function defined on $A \times M$, where M is an infinite subset of the Hausdorff space satisfying the second axiom of countability such that the closure of M is countably compact. Further, let $\sup F(H)$ be the least upper bound of the set $\{F(u) \mid u \in H\}$.

It is known that the limit superior as well as the least upper bound of the sequence of measurable functions are measurable functions.

Rożycki has proved in [2] the following lemma.

L e m m a 1. If for every $x \in A$ the function $F(x,\cdot)$ is upper semi-continuous in M and for every $t \in M$ the function $F(\cdot,t)$ is measurable in A and if we denote

$$\sup F(\cdot,M)(x) = \sup F(x,M),$$

$$\lim_{t \to a} \sup F(\cdot,t)(x) = \lim_{t \to a} \sup F(x,t).$$

then the functions $\sup F(\cdot,M)$, $\lim\limits_{t \to a} \sup F(\cdot,t)$ are measurable for every $a \in M'$, where M' is the derivative of the set set M.

Now we prove the theorem announced in the introduction.

T h e o r e m 3. If

1) for every $t \in M$ the function $F(\cdot,t)$ is measurable on A,
2) for every $x \in A$ the function $F(x,\cdot)$ is upper semi-continuous in M,
3) the limit $G(x) = \lim\limits_{t \to a} F(x,t)$ exists and is finite almost everywhere in A,

then for every $\varepsilon > 0$ there exists a set $B \subset A$ such that $\operatorname{mes}(A \setminus B) < \varepsilon$ and $F(x,t)$ tends upper uniformly on the set B to $G(x)$, when $t \to a$.

P r o o f. We may assume that outside a set of measure zero we have everywhere in A

$$\lim_{t \to a} F(x,t) = G(x)$$

and $G(x)$ is finite in A.

Let $\{U_n\}$ be a countable basis of the space X in the point a such that $U_{n+1} \subset U_n$ for $n \in N$. We put

$$F_n(x) = \sup F(x, U_n).$$

By lemma 1, the functions $F_n(x)$ are measurable on A. Since $\{U_n\}$ is a decreasing sequence of sets, the sequence $\{F_n\}$ is a non-increasing sequence of functions, therefore the limit $\lim F_n(x)$ exists.

It is easy to show that

$$\lim_{t \to a} F(x,t) = \lim_{t \to a} \sup F(x,t) = \lim_{n} F_n(x).$$

In virtue of assumption 3) the function $G(x)$ is measurable and finite. Since.

$$F_n(x) \geqslant F_{n+1}(x) \quad \text{and} \quad F_n(x) \to G(x),$$

by putting

$$C_n = \left\{ x : |F_n(x) - G(x)| < 1 \right\},$$

we obtain that $A = \bigcup_n C_n$ and $\operatorname{mes}(A) = \lim_{n \to \infty} \operatorname{mes}(C_n)$. Thus for every positive number $\frac{\varepsilon}{2}$ there exists n^* such that $\operatorname{mes}(A \setminus C) < \frac{\varepsilon}{2}$ and $F_n(x) < G(x) + 1$ for $n > n^*$, therefore the functions F_n $(n \in N)$ are finite on C.

By applying Egoroff's theorem to a subsequence $\{F_{n^*+k}\}$ we can find a set $B \subset C$ such that $m(C \setminus B) < \frac{\varepsilon}{2}$ and the sequence $\{F_{n+k}\}$ converges uniformly on B to G(x). We shall show that, when $t \to a$, the function F(x,t) converges upper uniformly on B to G(x). By definition, $F_n(x) = \sup F(x,U_n)$, where $U_n \supset U_{n+1}$. Thus for every positive ε there exists U_n such that

$$F_n(x) - G(x) < \varepsilon \qquad \text{i.e.} \quad \sup \; F(x,U_n) - G(x) < \varepsilon.$$

Therefore, if $t \in U_n$, then $F(x,t) - G(x) < \varepsilon$, q.e.d.

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