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SOME ALMOST ANALYTIC VECTOR FIELDS IN (f, g, u, v, λ)-STRUCTURE MANIFOLD

0. Introduction

The idea of almost analytic vector fields in an almost complex manifold has been studied by many mathematicians including Tachibana [3], Sasaki [4], and Yano [1]. Recently Hideaki Suzuki [2] obtained some results in the product manifold $M^{2n} \times R^2$ where M^{2n} is an (f, g, u, v, λ)-structure manifold [1] and R^2 is an 2-dimensional Euclidean plane. In the present paper, we define some vector fields in this product manifold and obtain necessary and sufficient conditions for these vector fields to be almost analytic. Later on, various particular cases have been studied.

1. Preliminaries.

Let M^{2n} be a 2n-dimensional differentiable manifold equipped with a (1,1) tensor field f, two vector fields U, V and two 1-forms u, v, such that*

$$f^2 = -1 + u \otimes U + v \otimes V \; ; \tag{1.1}$$

$$u \circ f = \lambda v; \quad fU = -\lambda V \; ; \tag{1.2}$$

*Throughout the paper all the tensor fields considered shall be C^∞.

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$$(1.3) \qquad v \circ f = -\lambda u \; ; \quad fV = \lambda U \; ;$$

$$(1.4) \qquad u(U) = 1 - \lambda^2 \; ; \quad u(V) = 0 \; ;$$

$$(1.5) \qquad v(V) = 1 - \lambda^2 \; ; \quad v(U) = 0 \; ;$$

1 being the unit tensor field while λ is a function. Such a manifold is called an (f, U, V, u, v, λ)- structure manifold and it is necessarily even dimensional. If an even dimensional manifold admits a positive definite Riemannian metric g satisfying the conditions

$$(1.6) \qquad g(U,X) = u(X); \quad g(V,X) = v(X);$$

$$g(fX,fY) = g(X,Y) - u(X)u(Y) - v(X)v(Y),$$

for any vector fields X and Y in M^{2n}, then we say M^{2n} is an (f, g, u, v, λ) - structure manifold.

We consider the product manifold $M^{2n} \times R^2$. Let the coordinates of an open neighbourhood $U_1 \subset M^{2n}$ be given by x^i * and the Cartesian co-ordinates of the plane by x^{∞_1}, x^{∞_2}, then $(x^i, x^{\infty_1}, x^{\infty_2})$ can be considered as the co-ordinates in the neighbourhood $(U \times R^2)$ in $M^{2n} \times R^2$.

Let U_i and U_1' be two coordinate neighbourhoods in M^{2n} such that $U_1 \cap U_i' \neq \emptyset$, then the intersection of the corresponding coordinate neighbourhoods $U_1 \times R^2$ and $U_1' \times R^2$ in the product manifold is also non-empty. Corresponding to the transformation

$$x^{i'} = x^{i'}(x^1, x^2, \ldots, x^{2n})$$

in $U_1 \cap U_1'$, we define the coordinate transformation in $(U_1 \times R^2) \cap (U_1' \times R^2)$ by

* Throughout this paper the indices i,j,k, ... run from 1 to 2n and A,B,C,... run from 1 to 2n+2.

$$x^{i'} = x^{i'}(x^i)\ ,$$

$$x^{\infty_1'} = x^{\infty_1}\ ,$$

$$x^{\infty_2'} = x^{\infty_2}\ ,$$

whose Jacobian matrix $\left(\frac{\partial x^{A'}}{\partial x^A}\right)$ is given by

$$\left(\frac{\partial x^{A'}}{\partial x^A}\right) = \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We define in the product manifold a (1,1) tensor field F whose local components are given by [1]

$$(F_B{}^A) = \begin{pmatrix} f_i{}^i & u^i & v^i \\ -u_j & 0 & -\lambda \\ -v_j & & 0 \end{pmatrix} .$$

It is easy to show that F is an almost complex structure. Moreover the product manifold admits a positive definite Riemannian metric G with local components

$$(G_{AB}) = \begin{pmatrix} g_{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

g_{ij} being the local components of the metric g in M^{2n}. It can be easily proved that G is an Hermitian metric.

Let $\left\{ {h \atop i\ j} \right\}$ and $\left[{A \atop B\ C} \right]$ be the Christoffel symbols corresponding to the Riemannian metrics g and G respectively and let ∇ and $\dot{\nabla}$ denote the respective covariant differentiation. After simple calculation, we get

$$\left\{ {h \atop i\ \ j} \right\} = \left[{h \atop i\ \ j} \right]$$

while the remaining components are zero. The covariant differentiation with respect to $\dot{\nabla}$ of the (1,1) tensor field F is by definition

$$\dot{\nabla}_C F_B{}^A = \frac{\partial F_B{}^A}{\partial x^C} + F_B{}^D \left[{A \atop D\ \ C} \right] - F_E{}^A \left[{E \atop B\ \ C} \right].$$

Thus the various components of ($\dot{\nabla}$ F) are given by

$$\dot{\nabla}_i F_k{}^j = \nabla_i f_k{}^j \; ; \qquad \nabla_i F_{\infty_1}{}^j = \nabla_i u^j \; ,$$

$$\dot{\nabla}_i F_{\infty_2}{}^j = \nabla_i v^j \; ; \qquad \nabla_i F_k{}^{\infty_1} = -\nabla_i u_k \; ,$$

$$\dot{\nabla}_i F_k{}^{\infty_2} = -\nabla_i v_k ; \quad \nabla_i F_{\infty_1}{}^{\infty_2} = -\nabla_i F_{\infty_2}{}^{\infty_1} = \nabla_i \lambda$$

and the remaining components are all zero.

2. Vector fields in the product manifold

Let X be any vector field in the base manifold M^{2n} with local components X^i, then we define its C - extension ${}^C(X)$ in the product manifold to be a vector field whose local components are given by

$$(2.1) \qquad \left({}^C(X)\right)^A = \begin{pmatrix} X^i \\ X^i u_i \\ X^i v_i \end{pmatrix}.$$

We also define three more vector fields in $M^{2n} \times R^2$ by

$$(2.2) \qquad \left({}^P(X)\right)^A = \begin{pmatrix} X^i \\ X^i u_i \\ 0 \end{pmatrix}; \qquad \left({}^Q(X)\right)^A = \begin{pmatrix} X^i \\ 0 \\ X^i v_i \end{pmatrix};$$

$$\left({}^H(X)\right)^A = \begin{pmatrix} X^i \\ 0 \\ 0 \end{pmatrix}.$$

These vector fields will be called P-extension, Q-extension and H-extension of the vector field X respectively.

Any vector field Z in the product manifold is called almost analytic iff

$$\left(\underset{Z}{\mathcal{L}}F\right)_B{}^A = Z^D \dot{\nabla}_D F_B{}^A - F_B{}^D \dot{\nabla}_D Z^A + F_D{}^A \dot{\nabla}_B Z^D = 0.$$

In case $Z = {}^C(X)$, then we have the following components,

$$
(2.3)\begin{cases}
\left(C\,\underset{(X)}{\alpha F}\right)^{i}_{j} = \left(\underset{X}{\alpha f}\right)^{i}_{j} + u^{i}\left(\underset{X}{\alpha u}\right)_{j} + v^{i}\left(\underset{X}{\alpha v}\right)_{j} + \\
\qquad + u^{i}X^{k}u_{jk} + v^{i}X^{k}v_{jk} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{i}_{\infty_1} = \left(\underset{X}{\alpha u}\right)^{i} \; ; \quad \left(C\,\underset{(X)}{\alpha F}\right)^{i}_{\infty_2} = \left(\underset{X}{\alpha v}\right)^{i} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_1}_{i} = f_{i}{}^{k}X^{e}u_{ek} - f_{i}{}^{k}\left(\underset{X}{\alpha u}\right)_{k} - \left(\underset{X}{\alpha u}\right)_{i} + \\
\qquad + \lambda X^{k}v_{ki} - \lambda\left(\underset{X}{\alpha v}\right)_{i} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_1}_{\infty_1} = -\, u^{i}\left(\underset{X}{\alpha u}\right)_{i} + u^{i}X^{k}u_{ki} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_1}_{\infty_2} = -\, X^{i}v_{i}\lambda - v^{i}\left(\underset{X}{\alpha u}\right)_{i} + v^{i}X^{k}u_{ki} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_2}_{i} = -\left(\underset{X}{\alpha v}\right)_{i} + \lambda\left(\underset{X}{\alpha u}\right)_{i} - \lambda X^{k}u_{ki} + \\
\qquad + f_{i}{}^{j}X^{k}v_{kj} - f^{k}{}_{i}\left(\underset{X}{\alpha v}\right)_{k} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_2}_{\infty_1} = X^{i}v_{i}\lambda - v^{i}\,\underset{X}{\alpha u}{}_{i} + v^{i}X^{k}u_{ki} \; ; \\
\left(C\,\underset{(X)}{\alpha F}\right)^{\infty_2}_{\infty_2} = -\, v^{i}\left(\underset{X}{\alpha v}\right)_{i} + u^{i}X^{k}v_{ki} \, .
\end{cases}
$$

These components are related by the following relations

$$
(2.4)\quad\left\{
\begin{aligned}
&\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_j^{\;i} u_i f_e^{\;j} = u_i u_e \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;i} + \\
&\qquad + u_i v_e \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;i} - (1-\lambda^2)\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_e^{\;\infty_1} - \lambda\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_e^{\;i} v_i ;\\
&\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_j^{\;i} v_i f_e^{\;j} = u_e v_i \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;i} + v_e v_i \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;i} + \\
&\qquad + \lambda\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_e^{\;i} u_i - (1-\lambda^2)\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_e^{\;\infty_2} ;\\
&\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_j^{\;i} u_i u^j = -\lambda v_j \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;j} - \lambda u_j \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;j} - \\
&\qquad - (1-\lambda^2)\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;\infty_1} ;\\
&\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_j^{\;i} v_i v^j = \lambda u_j \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;i} + \lambda\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;i} v_i - \\
&\qquad - (1-\lambda^2)\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;\infty_2} ;\\
&\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_j^{\;i} u_i v^j = \lambda u_i \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_1}^{\;i} - \lambda v_j \left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;j} - \\
&\qquad - (1-\lambda^2)\left(\underset{(X)}{\overset{\alpha}{C}} F\right)_{\infty_2}^{\;\infty_1} ;
\end{aligned}
\right.
$$

$$(2.4)\quad \left\{ \begin{aligned} \left(C\underset{(X)}{\overset{\alpha}{\mathcal{L}}}F\right)^{i}_{j} v_i u^j = \lambda u_j \left(C\underset{(X)}{\overset{\alpha}{\mathcal{L}}}F\right)^{j}_{\infty_1} - \lambda v_j \left(C\underset{(X)}{\overset{\alpha}{\mathcal{L}}}F\right)^{j}_{\infty_2} - \\ - (1-\lambda^2)\left(C\underset{(X)}{\overset{\alpha}{\mathcal{L}}}F\right)^{\infty}_{\infty} . \end{aligned} \right.$$

Throughout the paper, we shall assume that $(1-\lambda^2) \neq 0$, and

$$u_{ij} \stackrel{def}{=} \nabla_i u_j - \nabla_j u_i ;$$

$$v_{ij} \stackrel{def}{=} \nabla_i v_j - \nabla_j v_i .$$

T h e o r e m 2.1. A necessary and sufficient condition for the vector field $\overset{C}{(x)}$ in the product manifold to be almost analytic is

$$\left(\underset{X}{\overset{\alpha}{\mathcal{L}}}f\right)^{i}_{j} + u^i\left(\underset{X}{\overset{\alpha}{\mathcal{L}}}u\right)_j + v^i\left(\underset{X}{\overset{\alpha}{\mathcal{L}}}v\right)_j = u^i X^k u_{kj} + v^i X^k v_{kj} ,$$

$$\left(\underset{X}{\overset{\alpha}{\mathcal{L}}}U\right) = 0; \quad \left(\underset{X}{\overset{\alpha}{\mathcal{L}}}V\right) = 0.$$

P r o o f. The proof follows from (2.3) and (2.4).

C o r o l l a r y 2.1. The vector field ${}^{C}(U)$ is almost analytic in the product manifold iff

(i) $$\left(\overset{\alpha f}{U}\right)_j^{\;i} = 2\lambda\, u^i \nabla_j \lambda \;;$$

(ii) $$[U,V] = 0.$$

C o r o l l a r y 2.2. The vector field ${}^{C}(V)$ is almost analytic in the product manifold iff

(i) $$\left(\overset{\alpha f}{V}\right)_j^{\;i} = 2\lambda\, v^i \nabla_j \lambda \;;$$

(ii) $$[U,V] = 0.$$

In case $Z = {}^{P}(X)$, the components of $\left(P\overset{\alpha F}{(X)}\right)$ are given by

$$(2.5)\begin{cases} \left(P\overset{\alpha F}{(X)}\right)_j^{\;i} = \left(\overset{\alpha f}{X}\right)_j^{\;i} + u^i\left(\overset{\alpha u}{X}\right)_j + u^i X^k\, u_{jk} \;; \\ \left(P\overset{\alpha F}{(X)}\right)_{\infty_1}^{\;i} = \left(\overset{\alpha u}{X}\right)^i;\ \left(P\overset{\alpha F}{(X)}\right)_{\infty_2}^{\;i} = \left(\overset{\alpha v}{X}\right)^i \;; \\ \left(P\overset{\alpha F}{(X)}\right)_i^{\;\infty_1} = -\left(\overset{\alpha u}{X}\right)_i - f_i^{\;k}\, X^u{}_k - f_i^{\;k} X^e u_{ke} \;; \\ \left(P\overset{\alpha F}{(X)}\right)_{\infty_1}^{\;\infty_1} = -\, u^i\left(\overset{\alpha u}{X}\right)_i + u^i X^k u_{ki} \;; \end{cases}$$

$$
(2.5)\quad
\begin{cases}
\Big(P^{\alpha}_{(X)}F\Big)^{\infty_1}_{\infty_2} = - X^i \nabla_i \lambda - v^i \Big(\underset{X}{\overset{\alpha}{u}}\Big)_i + v^i X^k u_{ki}; \\[2ex]
\Big(P^{\alpha}_{(X)}F\Big)^{\infty_2}_{i} = - \Big(\underset{X}{\overset{\alpha}{v}}\Big)_i + \lambda \Big(\underset{X}{\overset{\alpha}{u}}\Big)_i - \lambda u_{ki} X^k ; \\[2ex]
\Big(P^{\alpha}_{(X)}F\Big)^{\infty_2}_{\infty_1} = X^i \nabla_i \lambda ; \\[2ex]
\Big(P^{\alpha}_{(X)}F\Big)^{\infty_2}_{\infty_2} = 0 .
\end{cases}
$$

These components satisfy the following relations

$$
(2.6)\quad
\begin{cases}
\Big(P^{\alpha}_{(X)}F\Big)^{i}_{j} u_i f_e{}^j = - (1-\lambda^2)\Big(P^{\alpha}_{(X)}F\Big)^{\infty_1}_{e} - \Big(P^{\alpha}_{(X)}F\Big)^{i}_{e} v_i + \\[1ex]
\qquad + u_i u_e \Big(P^{\alpha}_{(X)}F\Big)^{i}_{\infty_1} + u_i v_e \Big(P^{\alpha}_{(X)}F\Big)^{i}_{\infty_2} ; \\[2ex]
\Big(P^{\alpha}_{(X)}F\Big)^{i}_{j} u_i u^j = -(1-\lambda^2) \Big(P^{\alpha}_{(X)}F\Big)^{\infty_1}_{\infty_1} - \lambda u_i \Big(P^{\alpha}_{(X)}F\Big)^{i}_{\infty_2} - \\[1ex]
\qquad - \lambda v_i \Big(P^{\alpha}_{(X)}F\Big)^{i}_{\infty_1} ; \\[2ex]
\Big(P^{\alpha}_{(X)}F\Big)^{i}_{j} u_i v^j = -(1-\lambda^2)\Big(P^{\alpha}_{(X)}F\Big)^{\infty_1}_{\infty_2} +
\end{cases}
$$

$$
(2.6)\quad \begin{cases}
\quad + \lambda u_i \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_1}^{\ i} - \lambda v_i \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_2}^{\ i} ; \\
\left(P_{\overset{\alpha}{(X)}}F\right)_j^{\ i} v_i u^j = (1-\lambda^2)\left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_1}^{\ \infty_2} + \lambda u_j \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_1}^{\ j} = \\
\qquad - \lambda v_j \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_2}^{\ j} ; \\
\left(P_{\overset{\alpha}{(X)}}F\right)_j^{\ i} v_i f_e^{\ i} = -(1-\lambda^2)\left(P_{\overset{\alpha}{(X)}}F\right)_e^{\ \infty_2} + \lambda \left(P_{\overset{\alpha}{(X)}}F\right)_e^{\ i} u_i + \\
\qquad + u_e v_i \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_1}^{\ i} + v_e v_i \left(P_{\overset{\alpha}{(X)}}F\right)_{\infty_2}^{\ i} .
\end{cases}
$$

These relations yield the following theorems.

T h e o r e m 2.2. A necessary and sufficient condition for the P-extension ${}^{P}(X)$ of the vector field X in the base manifold to be almost analytic in the product manifold is

$$
\left(\underset{X}{\alpha} f\right)_j^{\ i} + u^i \left(\underset{X}{\alpha} u\right)_j + u^e X^k u_{jk} = 0;
$$

$$
\left(\underset{X}{\alpha} U\right) = 0 ; \qquad \left(\underset{X}{\alpha} V\right) = 0 .
$$

C o r o l l a r y 2.3. The vector field ${}^{P}(U)$ is almost analytic in the product manifold iff

$$
\left(\underset{U}{\alpha f}\right)_j^{\ i} = 2\lambda u^i v_j \lambda ; \quad U,V = 0.
$$

C o r o l l a r y 2.4. The vector field ${}^{P}(V)$ is almost analytic in the product manifold iff

$$
\left(\underset{V}{\alpha f}\right) = 0, \quad [U,V] = 0.
$$

In case $Z = {}^{Q}(X)$, then the components of $\left(Q \underset{(X)}{\alpha} F\right)$ are given by

$$
(2.7)\quad \begin{cases}
\left(Q \underset{(X)}{\alpha} F\right)_j^{\ i} = \left(\underset{X}{\alpha} f\right)_j^{\ i} + v^i\left(\underset{X}{\alpha} v\right)_j + v^i X^k v_{jk} ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_{\infty_1}^{\ i} = \left(\underset{X}{\alpha} u\right)^i ; \quad \left(Q \underset{(X)}{\alpha} F\right)_{\infty_2}^{\ i} = \left(\underset{X}{\alpha} v\right)^i ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_{\infty_1}^{\ \infty_1} = 0; \quad \left(Q \underset{(X)}{\alpha} F\right)_{\infty_2}^{\ \infty_1} = - X^i \nabla_i \lambda ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_i^{\ \infty_1} = - \left(\underset{X}{\alpha} u\right)_i - \lambda \left(\underset{X}{\alpha} v\right)_i - \lambda X^k v_{ik} ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_i^{\ \infty_2} = - \left(\underset{X}{\alpha} v\right)_i - f_i^{\ k} \left(\underset{X}{\alpha} v\right)_k - f_i^{\ k} X^e v_{ke} ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_{\infty_1}^{\ \infty_2} = X^i \nabla_i \lambda - u^i X^k v_{ik} - u^i \left(\underset{X}{\alpha} v\right)^i ; \\[2ex]
\left(Q \underset{(X)}{\alpha} F\right)_{\infty_2}^{\ \infty_2} = v^i \left(\underset{X}{\alpha} v\right)_i + v^i X^k v_{ik} .
\end{cases}
$$

These components satisfy the following relations

$$
\left(Q \underset{(X)}{\alpha} F\right)_j^{\ i} u_i f_e^j = -(1-\lambda^2)\left(Q \underset{(X)}{\alpha} F\right)_e^{\ \infty_1} + u_e u_i \left(Q \underset{(X)}{\alpha} F\right)_{\infty_1}^{\ i} +
$$

$$
(2.8)\quad \left\{
\begin{aligned}
&\qquad + u_e v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;i} - \lambda \left(\underset{(X)}{Q}^{\alpha} F\right)_{e}^{\;\;j} v_j \;; \\
&\left(\underset{(X)}{Q}^{\alpha} F\right)_{j}^{\;\;i} v_i f_e^j = - (1-\lambda^2)\left(\underset{(X)}{Q}^{\alpha} F\right)_{e}^{\;\;\infty_2} + \\
&\qquad + u_e v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_1}^{\;\;i} + v_e v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;i} + \\
&\qquad + v_e v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;i} + \lambda \left(\underset{(X)}{Q}^{\alpha} F\right)_{e}^{\;\;i} u_i \;; \\
&\left(\underset{(X)}{Q}^{\alpha} F\right)_{j}^{\;\;i} v^j v_i = \lambda u_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;i} + \lambda v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_1}^{\;\;i} - \\
&\qquad - (1-\lambda^2)\left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;\infty_2} \;; \\
&\left(\underset{(X)}{Q}^{\alpha} F\right)_{j}^{\;\;i} u_i v^j = \lambda u_j \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_1}^{\;\;j} - \lambda v_j \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;j} - \\
&\qquad - (1-\lambda^2)\left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;\infty_1} ; \\
&\left(\underset{(X)}{Q}^{\alpha} F\right)_{j}^{\;\;i} v_i u^j = \lambda u_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_1}^{\;\;i} - \lambda v_i \left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_2}^{\;\;i} - \\
&\qquad -(1-\lambda^2)\left(\underset{(X)}{Q}^{\alpha} F\right)_{\infty_1}^{\;\;\infty_2} .
\end{aligned}
\right.
$$

In view of the relations (2.7) and (2.8), we have the following theorem.

T h e o r e m 2.3. The Q-extension of the vector field X in the base manifold is almost analytic in the product manifold iff

$$\left(\underset{X}{\mathcal{L}} f\right)_j^{\ i} + v^i\left(\underset{X}{\mathcal{L}} v\right)_j + v^i X^k v_{jk} = 0; \quad \left(\underset{X}{\mathcal{L}} U\right) = 0; \quad \left(\underset{X}{\mathcal{L}} V\right) = 0 .$$

C o r o l l a r y 2.5. The Q-extension of the vector field U in the base manifold is almost analytic in the product manifold iff

$$\left(\underset{U}{\mathcal{L}} f\right) = 0; \quad [U,V] = 0.$$

C o r o l l a r y 2.6. A necessary and sufficient condition for the vector field ${}^{Q}(V)$ to be almost analytic in the product manifold is

$$\left(\underset{V}{\mathcal{L}} f\right)_j^{\ i} = 2v^i \lambda \nabla_j \lambda, \qquad [U,V] = 0.$$

Let $Z = {}^{H}(X)$, then the components of $\left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)$ are given by

$$(2.9)\quad \begin{cases} \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_j^{\ i} = \left(\underset{X}{\mathcal{L}} f\right)_j^{\ i}; \quad \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_1}^{\ i} = \left(\underset{X}{\mathcal{L}} u\right)^i, \\ \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_2}^{\ i} = \left(\underset{X}{\mathcal{L}} v\right)^i; \quad \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_i^{\ \infty_1} = -\left(\underset{X}{\mathcal{L}} u\right)_i, \\ \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_1}^{\ \infty_1} = 0; \quad \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_2}^{\ \infty_1} = -\left(\underset{X}{\mathcal{L}} \lambda\right), \\ \left(\underset{{}^{X}(X)}{\mathcal{L}} F\right)_i^{\ \infty_2} = -\left(\underset{X}{\mathcal{L}} v\right)_i; \quad \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_1}^{\ \infty_2} = \left(\underset{X}{\mathcal{L}} \lambda\right), \left(\underset{{}^{H}(X)}{\mathcal{L}} F\right)_{\infty_2}^{\ \infty_2} = 0. \end{cases}$$

These relations give the following

$$
(2.10)\quad
\begin{cases}
\left(\underset{H(X)}{\propto} F\right)_j^{\,i} u_i f_e^{\,j} = (1-\lambda^2)\left(\underset{H(X)}{\propto} F\right)_e^{\,\infty_1} + \\
\quad + u_i u_e \left(\underset{H(X)}{\propto} F\right)_{\infty_1}^{\,i} + u_i v_e \left(\underset{H(X)}{\propto} F\right)_{\infty_2}^{\,i} - \lambda v_j \left(\underset{H(X)}{\propto} F\right)^{j}; \\
\left(\underset{H(X)}{\propto} F\right)_j^{\,i} v_i f_e^{\,j} = -(1-\lambda^2)\left(\underset{H(X)}{\propto} F\right)_e^{\,\infty_2} + \\
\quad + v_i u_e \left(\underset{H(X)}{\propto} F\right)_{\infty_1}^{\,i} + v_i v_e \left(\underset{H(X)}{\propto} F\right)_{\infty_2}^{\,i} + \lambda u_j \left(\underset{H(X)}{\propto} F\right)_e^{\,j}; \\
\left(\underset{H(X)}{\propto} F\right)_j^{\,i} u_i v^j = \lambda u_i \left(\underset{H(X)}{\propto} F\right)_{\infty_1}^{\,i} - \lambda v_j \left(\underset{H}{\propto} F\right)_{\infty_2}^{\,j} - \\
\qquad - (1-\lambda^2)\left(\underset{H(X)}{\propto} F\right)_{\infty_2}^{\,\infty_1}.
\end{cases}
$$

The relations (2.9) and (2.10) yield the following

T h e o r e m 2.4. The H-extension ${}^{H}(X)$ of the vector field X in the base manifold is almost analytic in the product manifold iff

$$\left(\underset{X}{\propto} f\right) = 0; \quad \left(\underset{X}{\propto} U\right) = 0; \quad \left(\underset{X}{\propto} V\right) = 0.$$

C o r o l l a r y 2.7. A necessary and sufficient condition for the vector field ${}^{H}(U)$ to be almost analytic in the product manifold is

$$(\underset{U}{\alpha} f) = 0 \quad \text{and} \quad [U,V] = 0.$$

C o r o l l a r y 2.8. The vector field ${}^{H}(V)$ is almost analytic in the product manifold iff

$$(\underset{V}{\alpha} f) = 0; \quad [U,V] = 0.$$

We now study certain particular cases of the product manifold $M^{2n} \times R^2$.

3. Case 1. $M^{2n} \times R^2$ is a Kählerian manifold

The product manifold $M^{2n} \times R^2$ is Kählerian iff $(\dot{\nabla} F) = 0$. Thus

$$(3.1) \quad \begin{cases} 0 = \dot{\nabla}_k F_j{}^i = \nabla_k f_j{}^i; \quad 0 = \dot{\nabla}_k F_j{}^{\infty_1} = -\nabla_k u_j, \\ 0 = \dot{\nabla}_k F_j{}^{\infty_2} = -\nabla_k v_j; \quad 0 = \dot{\nabla}_k F^i = \nabla_k u^i, \\ 0 = \dot{\nabla}_k F_{\infty_2}{}^i = \nabla_k v^i; \quad 0 = \dot{\nabla}_k F_{\infty_2}{}^{\infty_1} = \dot{\nabla}_k F_{\infty_1}{}^{\infty_2} = -\nabla_k \lambda. \end{cases}$$

T h e o r e m 3.1. The P-extension ${}^{P}(X)$ of the vector field X in the base manifold is almost analytic in the product Kählerian manifold iff

$$(3.2) \quad f_k{}^e \nabla_e X^i - f_e{}^i \nabla_k X^e = 0,$$

$$(3.3) \quad (u^i u_p + v^i v_p) \nabla_k X^p = 0.$$

P r o o f: Let the conditions (3.2) and (3.3) hold. Operating v_i on (3.3), we get

$$(1-\lambda^2)v_p \nabla_k X^p = 0 ,$$

i.e.

(3.4) $$v_p \nabla_k X^p = 0 .$$

Again operating u_i on (3.3), we have

(3.5) $$u_p \,_k X^p = 0 .$$

The transvection of (3.2) with $f_j^{\ k}$ gives

(3.6) $$\nabla_j X^i + f_j^{\ k} f_e^{\ i} \nabla_k X^e = 0 .$$

Operating (3.6) by v^j and u^j respectively, we have

(3.7) $$v^j \nabla_j X^p = 0 \quad \text{and} \quad u^j \nabla_j X^p = 0 .$$

In view of the results (3.4), (3.7), (3.3), (3.1) and Theorem 2.2, ${}^P(X)$ is almost analytic.

Conversely, let ${}^P(X)$ be almost analytic in Kählerian manifold then in view of Theorem 2.1, and (3.1), we get

$$\left(\underset{X}{\mathcal{L}} f\right)_j^{\ i} + u^i \left(\underset{X}{\mathcal{L}} u\right)_j = 0 ; \qquad \left(\underset{X}{\mathcal{L}} U\right) = 0 \quad \text{and} \quad \left(\underset{X}{\mathcal{L}} V\right) = 0 .$$

The relations (2.5) provide that

(3.8) $$\left(\underset{X}{\mathcal{L}} f\right)_j^{\ i} = 0 ,$$

i.e. $$X^k \nabla_k f_j^{\ i} - f_j^{\ k} \nabla_k X^i = 0 - f_k^{\ i} \nabla_j X^k .$$

Hence in view of (3.1)

(3.9) $$f_k^{\ i}\nabla_j X^k - f_j^{\ k}\nabla_k X^i = 0 .$$

Again from (2.5)

$$\left(P \underset{(X)}{\alpha} F\right)_i^{\infty} = -\left(\underset{X}{\alpha} v\right)_i + \lambda\left(\underset{X}{\alpha} u\right)_i - \lambda u_{ki} X^k$$

gives $\left(\underset{X}{\alpha} v\right)_i = 0,$ as $u^i\left(\underset{X}{\alpha} u\right)_j = 0$ by (3.8),

i.e.

(3.10) $$v^p v_k \nabla_i X^k = 0 .$$

The result $u^i\left(\underset{X}{\alpha} u\right) = 0$ gives

(3.11) $$u^i u_k \nabla_j X^k = 0 .$$

Addition of (3.10) and (3.11) yields the second condition (3.3). Hence the proof.

T h e o r e m 3.2. The vector field ${}^P(X)$ is almost analytic iff one of the vector fields ${}^Q(X)$ and ${}^H(X)$ is almost analytic.

P r o o f. Theorem 3.1 yields that ${}^P(X)$ is almost analytic iff

(a) $$f_i^{\ k}\nabla_k X^e - f_k^{\ e}\nabla_i X^k = 0$$

and

(b) $$(u^e u_p + v^e v_p)\nabla_j X^p = 0 .$$

The relation (a) gives

$$\left(\underset{X}{\alpha} f\right) = 0 ,$$

and (b) yields

$$\left(\underset{X}{\alpha} U\right) = 0 , \qquad \left(\underset{X}{\alpha} V\right) = 0$$

(with the help of (3.7)).

Hence, in view of Theorem 2.4, $^H(X)$ is almost analytic.
We also note that by (3.1) and (b) we have

$$v^i \left(\underset{X}{\alpha} v\right)_j = v^i v_k \nabla_j X^k = 0$$

Hence $$\left(\underset{X}{\alpha} f\right)_j{}^i + v^i \left(\underset{X}{\alpha} v\right)'_j = 0$$

and $$\left(\underset{X}{\alpha} U\right) = 0 ; \qquad \left(\underset{X}{\alpha} V\right) = 0 .$$

Thus in view of Theorem 2.3, the vector field $^Q(X)$ is almost analytic.

Conversely, let $^Q(X)$ be almost analytic, then in view of Theorem (2.3), relation (3.1) and 2.7, we have

$$\left(\underset{X}{\alpha} f\right) = 0 ; \qquad \left(\underset{X}{\alpha} U\right) = 0 ; \qquad \left(\underset{X}{\alpha} V\right) = 0$$

i.e. $$\left(\underset{X}{\alpha} f\right)_j{}^i = X^k \nabla_k f_j{}^i - f_j{}^k \nabla_x X^i + f_x{}^i \nabla_j X^k =$$

$$= f_k{}^i \nabla_j X^k - f_j{}^k \nabla_k X^i = 0$$

Again the fact $v^i\left(\underset{X}{\alpha}v\right)_j = 0$ gives

(c) $$v^i v_k \nabla_j X^k = 0$$

and the result (2.7) gives

$$\left(Q\underset{(X)}{\alpha}F\right)_i^{\infty_1} = \left(\underset{X}{\alpha}u\right)_i - \left(\underset{X}{\alpha}v\right)_i - X^k v_{ik} = 0$$

i.e. $\left(\underset{X}{\alpha}u\right)_i = 0;$ as $v^i\left(\underset{X}{\alpha}v\right)_j = 0$

i.e.

(d) $$u^k u_j \nabla_i X^j = 0 .$$

Hence (c) and (d) give

$$(u^k u_j + v^k v_j) \nabla_i X^j = 0 .$$

Thus the vector field ${}^P(X)$ is almost analytic. In case ${}^H(X)$ is almost analytic then it is very easy to prove that ${}^P(X)$ is also almost analytic.

T h e o r e m 3.3. The vector fields ${}^C(U)$, ${}^C(V)$, ${}^P(U)$, ${}^P(V)$, ${}^Q(U)$, ${}^Q(V)$, ${}^H(U)$ and ${}^H(V)$ are all almost analytic in the Kählerian manifold.

P r o o f. The proof follows from corollaries (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8) and from the relation (3.1).

4. Case 2. $M^{2n} \times R^2$ is an almost Kählerian manifold

The product manifold $M^{2n} \times R^2$ is an almost Kählerian iff

$$F_{ABC} = \dot{\nabla}_A F_{BC} + \dot{\nabla}_B F_{CA} + \dot{\nabla}_C F_{AB} = 0 ,$$

where $F_{BC} = F_B{}^A G_{AC}$.

The components of $F_{ABC} = 0$ are given by

$$
(4.1)\quad \begin{cases}
F_{ijh} = f_{ijh} = F_{jih} = F_{jhi} = \nabla_i f_{jh} + \nabla_j f_{hi} + \nabla_h f_{ij} = 0, \\
F_{\infty_1 jh} = F_{j\infty_1 h} = F_{jh\infty_1} = \nabla_j u_h - \nabla_h u_j = 0, \\
F_{\infty_2 jh} = F_{j\infty_2 h} = F_{jh\infty_2} = \nabla_j v_h - \nabla_h v_j = 0, \\
F_{\infty_1\infty_1 h} = F_{\infty_1 h\infty_1} = F_{\infty_1} F_{h\,1\infty_1} = 0 \\
F_{\infty_1\infty_2 h} = F_{\infty_2\infty_1 h} = F_{\infty_2 h\infty_1} = \nabla_h \lambda = 0, \\
F_{\infty_2\infty_2 h} = F_{\infty_2 h\infty_2} = F_{h\infty_2\infty_2} = 0, \\
F_{\infty_1\infty_1\infty_1} = F_{\infty_2\infty_2\infty_2} = F_{\infty_1\infty_2\infty_2} = F_{\infty_2\infty_1\infty_2} = F_{\infty_2\infty_2\infty_1} = 0.
\end{cases}
$$

T h e o r e m 4.1. The vector field ${}^C(X)$ is almost analytic in the almost Kählerian manifold iff

$$\left(\underset{X}{\mathcal{L}} f\right)_j{}^i + u^i\left(\underset{X}{\mathcal{L}} u\right)_j + v^i\left(\underset{X}{\mathcal{L}} v\right)_j = 0, \quad [U,V] = 0.$$

P r o o f. The theorem is an easy consequence of the theorem 2.1 and 4.1.

T h e o r e m 4.2. The vector field ${}^C(U)$ is almost analytic in almost Kählerian manifold iff

$$f_j^{\ i}\nabla_m u_i = 0, \ [U,V] = 0.$$

P r o o f. Theorem 4.1 and the relation (4.1) gives that $C(U)$ is almost analytic iff

$$\left(\mathop{\alpha f}_{U}\right)_j^{\ i} = 0 \ ; \ [U,V] = 0.$$

Now

$$\left(\mathop{\alpha f}_{U}\right)_j^{\ i} = u^k\nabla_k f_j^{\ i} - f_j^{\ k}\nabla_k u^i + \nabla_k u^i + f_k^{\ i}\nabla_j u^k \ ,$$

$$g_{im}(\alpha_U f)_j^{\ i} = u^k g_{im}\nabla_k f_j^{\ i} - f_j^{\ k} g_{im}\nabla_k u^i + f_k^{\ i} g_{im}\nabla_j u^k =$$

$$= u^k\nabla_k f_{jm} + f_{km}\nabla_j u^k - f_j^{\ k}\nabla_m u_k =$$

$$= u^k\nabla_k f_{jm} + \nabla_j u^k f_{km} - u^k\nabla_j f_{km} - f_j^{\ k}\nabla_m u_k =$$

$$= u^k(\nabla_k f_{jm} + \nabla_j f_{mk}) - \lambda\nabla_j v_m - f_j^{\ k}\nabla_m u_k =$$

$$= - u^k(\nabla_m f_{kj}) - \lambda\nabla_j v_m - f_j^{\ k}\nabla_m u_k =$$

$$= u_i\nabla_m f_j^{\ i} - \lambda\nabla_j v_m - \lambda\nabla_m v_j + u_k\nabla_m f_j^{\ k} =$$

$$= 2u_i\nabla_m f_j^{\ i} - \lambda(\nabla_j v_m + \nabla_m v_j) =$$

$$= 2u_i\nabla_m f_j^{\ i} - 2\lambda\nabla_m v_j =$$

$$- 2f_j^{\ i}\nabla_m u_i = 0.$$

Hence the proof.

Similarly, we have the following

T h e o r e m 4.3. The vector field ${}^{C}(V)$ is almost analytic in the almost Kählerian manifold iff

$$f_j{}^i \nabla_m v_i = 0, \ [U,V] = 0.$$

T h e o r e m 4.4. The vector field ${}^{C}(U)$ is almost analytic in the almost Kählerian manifold iff any one of the vector fields ${}^{P}(U)$, ${}^{Q}(U)$, ${}^{H}(U)$ is almost analytic in the almost Kählerian manifold.

P r o o f. The proof follows from Theorems 4.2, 2.3, 2.4, 2.5 and the relation 4.1.

5. Case 3. $M^{2n} \times R^2$ is nearly Kählerian

The product manifold $M^{2n} \times R^2$ is nearly Kählerian iff

$$\mathring{\nabla}_A F_B{}^C + \mathring{\nabla}_B F_A{}^C = 0.$$

Thus in a nearly Kählerian manifold, we have

$$(5.1) \quad \begin{cases} \nabla_i f_k{}^j + \nabla_k f_i{}^j = 0; \quad \nabla_i u_j + \nabla_j u_i = 0, \\ \nabla_i v_j + \nabla_j v_i = 0; \quad \nabla_i u^j = 0; \quad \nabla_i v^j = 0, \\ \nabla_i \lambda = 0. \end{cases}$$

Consequently

$$\nabla_i u_j = 0; \qquad \nabla_i v_j = 0.$$

T h e o r e m 5.1. In nearly Kählerian manifold $M^{2n} \times R^2$, the vector fields ${}^{C}(U)$, and ${}^{C}(V)$ are almost analytic.

P r o o f. In view of the theorem 2.1, ${}^{C}(X)$ is almost analytic iff

$$\left(\underset{X}{\mathcal{L}}f\right)_j{}^i + u^i\left(\underset{X}{\mathcal{L}}u\right)_j + v^i\left(\underset{X}{\mathcal{L}}v\right)_j = 0,$$

$$\left(\underset{X}{\mathcal{L}}U\right) = 0; \qquad \left(\underset{X}{\mathcal{L}}V\right) = 0.$$

In case X = U; then in view of (5.1), we have

$$\left(\underset{X}{\mathcal{L}}f\right)_j{}^i = u^k \nabla_k f_j{}^i - f_j{}^k \nabla_k u^i + f_k{}^i \nabla_j u^k =$$

$$= -u^k \nabla_j f_k{}^i = \lambda \nabla_j v^i + f_k{}^i \nabla_j u^k = 0.$$

Also,

$$\left(\underset{U}{\mathcal{L}}u\right) = 0; \qquad \left(\underset{U}{\mathcal{L}}v\right) = 0; \qquad \left(\underset{U}{\mathcal{L}}V\right) = 0.$$

Hence ${}^{C}(U)$ is almost analytic in the nearly Kählerian manifold. Similarly, we can prove that ${}^{C}(V)$ is almost analytic.

T h e o r e m 5.2. The vector field ${}^{P}(X)$ is almost analytic in nearly Kählerian manifold iff

$$\left(\underset{X}{\mathcal{L}}f\right)_j{}^i = -u^i u_k \nabla_j X^k,$$

$$u^k \nabla_k X^i = 0; \qquad v^k \nabla_k X^i = 0.$$

P r o o f. The proof follows from Theorem 2.2 and the result 5.1.

C o r o l l a r y 5.2. The vector fields ${}^{P}(U)$ and ${}^{P}(V)$ are almost analytic in the nearly Kählerian manifold.

T h e o r e m 5.3. The vector fields ${}^{Q}(U)$, ${}^{H}(U)$ are almost analytic in the nearly Kählerian manifold.

P r o o f. The vector field ${}^{Q}(X)$ is almost analytic in the nearly Kählerian manifold iff

$$\left(\underset{X}{\mathcal{L}}f\right)_j{}^i + v^i v_k \nabla_j X^k = 0,$$

$$u^k \nabla_k X^i = 0; \quad v^k \nabla_k X^i = 0.$$

Thus in case X = U we then have in view of (5.1)

$$\left(\underset{U}{\mathcal{L}}f\right)_j{}^i = u^k \nabla_k f_j{}^i = f_j{}^k \nabla_k u^i + f_k{}^i \nabla_j u^k = 0.$$

Hence ${}^{Q}(U)$ is almost analytic. Similarly, we can prove the vector field ${}^{H}(U)$ is almost analytic.

On the same lines, we can also prove the following

T h e o r e m 5.4. The vector fields ${}^{Q}(V)$ and ${}^{H}(V)$ are almost analytic in the nearly Kählerian manifold.

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