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ON THE SCHWARZIAN FOR THE FUNCTIONS OF SHAH-TAO-SHING

In this article we consider the class K_D of functions of Shah-Tao-shing. This class consists of functions f that are holomorphic and single-valued in a simply - connected domain D, $0 \in D$, normed by the condition $f(0) = 0$ and satisfying the condition

$$f(z_1)\,\overline{f(z_2)} \neq -1 \quad \text{for } z_1, z_2 \in D.$$

Necessary conditions are established in order that a function belong to K_D, similar to those given by Singh [1] for the class of bounded single-valued functions. In particular, the case $D = K(0,1)$ is considered.

1. Let B be a simply-connected domain having the property that if $w \in \bar{B}$, then $-\frac{1}{\bar{w}} \notin \bar{B}$ and let $0 \in B$. The boundary b of B is an analytic curve,

$$b:\ w = w(t), \qquad \alpha \leqslant t \leqslant \beta ,$$

which is positively oriented with respect to its interior. The curve

$$\tilde{b}: \; w = \tilde{w}(t) = -\frac{1}{\overline{w}(t)}, \; \alpha \leq t \leq \beta$$

has no points in common with $\bar{B}$. The curve $\tilde{b}$ is also positively oriented with respect to its interior, because the curves b and $\tilde{b}$ have the same indices with respect to the point $w = 0$. In fact,

$$\operatorname{ind}_b 0 = \frac{1}{2\pi i}\int_b \frac{dw}{w} = \frac{1}{2\pi}\Delta\arg w = 1 ,$$

$$\operatorname{ind}_{\tilde{b}} 0 = \frac{1}{2\pi i}\int_{\tilde{b}} dw = \frac{1}{2\pi i}\int_b \frac{-d\overline{w}}{\overline{w}} =$$

$$= \frac{-1}{2\pi}\Delta\arg\overline{w} = \frac{1}{2\pi}\Delta\arg w = 1 .$$

It is easy to see that if points $\zeta_1, \zeta_2, \ldots, \zeta_n$ belong to B, then the points $-\frac{1}{\overline{\zeta}_1}, -\frac{1}{\overline{\zeta}_2}, \ldots, -\frac{1}{\overline{\zeta}_n}$ lie outside the curve $\tilde{b}$. Indeed, if $\zeta_k \in B$, then

$$\operatorname{ind}_{\tilde{b}}\left(-\frac{1}{\overline{\zeta}_k}\right) = \frac{1}{2\pi i}\int_{\tilde{b}} \frac{dw}{w + \frac{1}{\overline{\zeta}_k}} = \frac{1}{2\pi i}\int_b \frac{-d\overline{w}}{\overline{w}} + \frac{1}{2\pi i}\int_b \frac{d\overline{w}}{\overline{w} - \overline{\zeta}_k} =$$

$$= 1 - 1 = 0.$$

We now consider the function

$$p(w) = \operatorname{Re}\sum_{\nu=1}^{n}\left[\alpha_\nu \frac{\partial g(w,\zeta)}{\partial\zeta}\bigg|_{\zeta=\zeta_\nu}\right], \tag{1}$$

where $g(w,\zeta)$ is the Green function for the domain B with a pole at ζ, $\zeta \in B$, and α_ν are arbitrary complex numbers.

The function p(w) is harmonic in B except for the points ζ_ν and it vanishes on the boundary b. The singular part of p(w) has the form

$$S_1(w) = \frac{1}{2}\,\mathrm{Re}\left\{\sum_{\nu=1}^{n} \frac{\alpha_\nu}{w-\zeta_\nu}\right\}.$$

If we put

(2) $$S(w) = -\frac{1}{2}\,\mathrm{Re}\left\{\sum_{\nu=1}^{n}\left(\frac{\alpha_\nu}{w-\zeta_\nu} - \frac{\bar{\alpha}_\nu w}{1+\bar{\zeta}_\nu w}\right)\right\},$$

then the function p(w) + S(w) is harmonic in B and p(w) = 0 for $w \in b$. Let us observe that

(3) $$S\left(-\frac{1}{\bar{w}}\right) = S(w).$$

Next let V(w) denote a meromorphic function such that

(4) $$\mathrm{Re}\left\{V(w)\right\} = S(w).$$

It is easy to verify that

(5) $$V(w) = \overline{V\left(-\frac{1}{\bar{w}}\right)}.$$

Let B^* be the region lying between the curves b and $\tilde{b}$. From Green's theorem it follows that

(6) $$0 \leq \iint_{B^*} |V'(w)|^2\, dw = \mathrm{Re}\left\{\frac{1}{i}\int_{-b+\tilde{b}} \overline{V(w)}\, dV(w)\right\} =$$

$$= \mathrm{Re}\left\{\frac{1}{i}\Big(2\int_{-b+\tilde{b}} \mathrm{Re}\left\{V(w)\right\}\, d\mathrm{Re}\left\{V(w)\right\} + 2i\int_{-b+\tilde{b}} \mathrm{Re}\left\{V(w)\right\} dI_m\left\{V(w)\right\} -$$

$$- \int\limits_{-b+\tilde{b}} V(w)dV(w))\Big\} = 2\int\limits_{-b} \mathrm{Re}\Big\{V(w)\Big\}d\,\mathrm{Im}\Big\{V(w)\Big\} + $$

$$+\int\limits_{\tilde{b}} \mathrm{Re}\Big\{V(w)\Big\}d\,\mathrm{Im}\Big\{V(w)\Big\}.$$

Putting

$$J_1 = \int\limits_{-b} \mathrm{Re}\Big\{V(w)\Big\}\ d\ \mathrm{Im}\Big\{V(w)\Big\},$$

$$J_2 = \int\limits_{\tilde{b}} \mathrm{Re}\Big\{V(w)\Big\}d\ \mathrm{Im}\Big\{V(w)\Big\}\ ,$$

and taking into account (4), (5) and (6) we obtain

$$J_1 = \int\limits_{-b} S(w)d\,\mathrm{Im}\Big\{V(w)\Big\} = -\int\limits_{-b} S\Big(-\frac{1}{\overline{w}}\Big)\,d\,\mathrm{Im}\Big\{\overline{V\Big(-\frac{1}{\overline{w}}\Big)}\Big\} =$$

$$= \int\limits_{\tilde{b}} S(w)d\,\mathrm{Im}\Big\{V(w)\Big\} = J_2\ .$$

Thus, the inequality (6) can be replaced by the following one

$$\int\limits_{-b}\mathrm{Re}\Big\{V(w)\Big\}d\,\mathrm{Im}\Big\{V(w)\Big\} \geq 0. \tag{7}$$

Let U(w) denote a function meromorphic within B for which

$$\mathrm{Re}\Big\{U(w)\Big\} = p(w)\ .$$

The function $U(w) + V(w)$ is holomorphic in B. Hence, by Green's theorem, it follows that

$$0 \leq \iint\limits_{B} \big|U'(w)+V'(w)\big|^2\ dw = \mathrm{Re}\Big\{\frac{1}{i}\int\Big\{\overline{U(w)+V(w)}\Big\}\ d\Big[U(w)+V(w)\Big]\Big\} =$$

$$= 2\int_b \mathrm{Re}\{U(w)+V(w)\}\, d\,\mathrm{Im}\{U(w)+V(w)\} =$$

$$= 2\int_b \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{U(w)\} + 2\int_b \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{V(w)\} .$$

From above and from (7) it follows that

$$\int_b \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{U(w)\} \geqslant \int_{-b} \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{V(w)\} \geqslant 0 ,$$

hence

(8) $$\int_b \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{U(w)\} \geqslant 0 .$$

Since on the curve b we have $p(w) = 0$, it follows that

(9) $$\int_b \mathrm{Re}\{V(w)\}\, d\,\mathrm{Im}\{U(w)\} = \int_b \mathrm{Re}\{V(w)+U(w)\}\cdot d\,\mathrm{Im}\{U(w)\}.$$

Now observe that

$$\int_b \{U(w)+V(w)\}\, dU(w) =$$

$$\int_b \left[\mathrm{Re}\{U(w)+V(w)\} + i\,\mathrm{Im}\{U(w)+V(w)\}\right]\cdot d\left[\mathrm{Re}\{U(w)\} + i\,\mathrm{Im}\{U(w)\}\right] =$$

$$= \int_b \mathrm{Re}\{U(w)+V(w)\}\, d\mathrm{Re}\{U(w)\} + i\int_b \mathrm{Im}\{U(w)+V(w)\}\, d\mathrm{Re}\{U(w)\} +$$

$$+ i\int_b \mathrm{Re}\{U(w)+V(w)\}\, d\,\mathrm{Im}\{U(w)\} - \int_b \mathrm{Im}\{U(w)+V(w)\}\, d\,\mathrm{Im}\{U(w)\} .$$

Taking into account that $\mathrm{Re}\{U(w)\} = p(w) = 0$ for $w \in b$, we obtain

$$\int_b \{U(w)+V(w)\}\, d\{U(w)\} =$$

$$= i\int_b \mathrm{Re}\{U(w)+V(w)\}\, d\,\mathrm{Im}\{U(w)\} - \int_b \mathrm{Im}\{U(w)+V(w)\}\, d\,\mathrm{Im}\{U(w)\}$$

from which it follows that

$$(10)\qquad \mathrm{Re}\left[\frac{1}{i}\int_b \{U(w)+V(w)\}\,dU(w)\right] = \int \mathrm{Re}\{U(w)+V(w)\}\,d\,\mathrm{Im}\{U(w)\}.$$

Since the function V(w) + U(w) is holomorphic in $\bar{B}$, we get

$$\int_b \{V(w)+U(w)\}\,d\{V(w)+U(w)\} = 0$$

and consequently

$$(11)\qquad \int\{V(w)+U(w)\}\,dU(w) = -\int\{V(w)+U(w)\}\,dV(w)\ .$$

In view of (8), (9), (10) and (11), we obtain

$$0 \leqslant \mathrm{Re}\left[\frac{1}{i}\int_b\{V(w)+U(w)\}\,dU(w)\right] = -\,\mathrm{Re}\left[\frac{1}{i}\int_b\{V(w)+U(w)\}\,dV(w)\right].$$

With the help of integration by parts, we finally obtain

$$(12)\qquad \mathrm{Re}\left\{\frac{1}{i}\int_b\left[U'(w)+V'(w)\right]V(w)\,dw\right\} \geqslant 0.$$

Next, observe that

$$(13)\qquad U(w) = \frac{1}{2}\sum_{\nu=1}^{n}\left[\alpha_\nu \frac{\partial q(w,\zeta)}{\partial \zeta}\Big|_{\zeta=\zeta_\nu} + \bar{\alpha}_\nu \frac{\partial q(w,\zeta)}{\partial \bar{\zeta}}\Big|_{\zeta=\zeta_\nu}\right],$$

where $q(w,\zeta)$ is a meromorphic function of w such that $\mathrm{Re}\{q(w,\zeta)\} = g(w,\zeta)$. It is easy to verify that

$$\frac{\partial q(w,\zeta)}{\partial w} = 2\,\frac{\partial g(w,\zeta)}{\partial w},\quad w \neq \zeta.$$

Taking into account that for $w \neq \zeta_\nu$

$$U'(w) = \frac{\partial U}{\partial w}$$

and in view of (13) we get

$$(14)\qquad U'(w) = \sum_{\nu=1}^{n}\left[\alpha_\nu \frac{\partial^2 g(w,\zeta)}{\partial \zeta_\nu \partial w}\bigg|_{\zeta=\zeta_\nu} + \bar{\alpha}_\nu \frac{\partial^2 g(w,\zeta)}{\partial \bar{\zeta}\partial w}\bigg|_{\zeta=\zeta_\nu}\right].$$

On the other hand, from (2) and (4) we obtain

$$V'(w) = -\frac{1}{2}\sum_{\nu=1}^{n}\left[-\frac{\alpha_\nu}{(w-\zeta_\nu)^2} - \frac{\bar{\alpha}_\nu}{(1+\bar{\zeta}_\nu w)^2}\right]$$

which together with (14) implies

$$(15)\qquad U'(w)+V'(w) = \sum_{\nu=1}^{n}\left[\alpha_\nu\left(\frac{\partial^2 g(w,\zeta)}{\partial\zeta\,\partial w}\bigg|_{\zeta=\zeta_\nu} + \frac{1}{2(w-\zeta_\nu)^2}\right) + \right.$$

$$\left. + \bar{\alpha}_\nu\left(\frac{\partial^2 g(w,\zeta)}{\partial\bar{\zeta}_\nu\partial w}\bigg|_{\zeta=\zeta_\nu} + \frac{1}{2(1+\bar{\zeta}_\nu w)^2}\right)\right].$$

Following Bergman [2], we denote

$$(16)\qquad K(w,\bar{\zeta}) = -\frac{2}{\pi}\,\frac{\partial^2 g(w,\zeta)}{\partial w\,\partial\bar{\zeta}},$$

$$(16')\qquad \chi(w,\zeta) = \frac{2}{\pi}\,\frac{\partial^2 g(w,\zeta)}{\partial\zeta\,\partial w} + \frac{1}{\pi(w-\zeta)}.$$

Inserting (16) and (16′) into (15) we obtain

$$U(w)+V(w) = \sum_{\nu=1}^{n}\left[\alpha_\nu\frac{\pi}{2}\chi(w,\zeta)+\bar{\alpha}_\nu\left(\frac{1}{2(1+\bar{\zeta}_\nu w)^2} - \frac{\pi}{2}K(w,\bar{\zeta}_\nu)\right)\right]$$

and consequently

$$\frac{1}{i}\int_b \left[U'(w)+V'(w)\right]V(w)dw = \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[K(\zeta_\nu, \bar{\zeta}_\mu) - \right.$$

$$\left. - \frac{1}{\pi(1+\zeta_\nu \bar{\zeta}_\mu)^2}\right] - \sum_{\nu,\mu=1}^{n} \alpha_\nu \alpha_\mu \chi(\zeta_\nu, \zeta_\mu) .$$

From the equality above and from (12) we obtain

$$(17) \quad \mathrm{Re} \left\{ \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[K(\zeta_\nu, \bar{\zeta}_\mu) - \frac{1}{\pi(1+\zeta_\nu \bar{\zeta}_\mu)^2}\right] - \right.$$

$$\left. - \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \chi(\zeta_\nu, \zeta_\mu) \right\} \geq 0 .$$

Hence, we have proved the following theorem.

T h e o r e m 1. If B is a simply-connected region with analytic boundary such that $w \in \bar{B}$ implies $-\frac{1}{\bar{w}} \notin \bar{B}$ and if $0 \in B$, then for any points $\zeta_1, \zeta_2, \ldots, \zeta_n$ in B there holds the inequality (17); $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary complex numbers and the functions $K(w,\bar{\zeta})$ and $\chi(w,\zeta)$ are defined by (16) and (16'), $g(w,\zeta)$ being the Green function for the domain B.

2. Let f(z) be a function which maps a simply-connected domain D conformally onto the domain B described in section 1. Then we have the well known relations between the functions K_D and χ_D for the domain D and the functions K_B and χ_B for the domain B

$$(18) \qquad K_D(z,\bar{\zeta}) = K_B(w,\bar{\omega})f'(z)\,\overline{f'(\zeta)},$$

$$(18') \qquad \chi_D(z,\zeta) = \chi_B(w,\omega)f'(z)\,f'(\zeta) - u(z,\zeta) ,$$

where

$$\omega = f(\zeta), \quad w = f(z), \quad u(z,\zeta) = \frac{1}{\pi}\left[\frac{f'(z)f'(\zeta)}{[f(z)-f(\zeta)]^2} - \frac{1}{(z-\zeta)^2}\right].$$

Let η_ν , $\nu = 1,2,\dots,n$, be arbitrary points of D and $\zeta_\nu = f(\eta_\nu)$ their images under the map f. Then inequality (17) takes the form

$$\text{(19)} \qquad \operatorname{Re}\left\{ \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[K(\eta_\nu, \eta_\mu) - \frac{f'(\eta_\nu)f'(\eta_\mu)}{\pi[1+f(\eta_\nu)\overline{f(\eta_\mu)}]^2}\right] - \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[\chi(\eta_\nu, \eta_\mu) + u(\eta_\nu, \eta_\mu)\right]\right\} \geq 0 .$$

Now, let us substitute $\alpha_\nu = \alpha'_\nu e^{-i\frac{\theta}{2}}$, where

$$\theta = \arg \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[K(\eta_\nu, \eta_\mu) - \frac{f'(\eta_\nu)\overline{f'(\eta_\mu)}}{\pi[1+f(\eta_\nu)\overline{f(\eta_\mu)}]^2}\right].$$

We obtain

$$\operatorname{Re}\left\{ \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[K(\eta_\nu, \eta_\mu) - \frac{f'(\eta_\nu)\overline{f'(\eta_\mu)}}{[1+f(\eta_\nu)\overline{f(\eta_\mu)}]^2}\right]\right\} =$$

$$= \operatorname{Re}\left\{e^{-i\theta} \left| \sum_{\nu,\mu=1}^{n} \alpha'_\nu \alpha'_\mu \left[K(\eta_\nu, \eta_\mu) - \frac{f'(\eta_\nu)\overline{f'(\eta_\mu)}}{\pi[1+f(\eta_\nu)\overline{f(\eta_\mu)}]^2}\right]\right| e^{i\theta}\right\} =$$

$$= \left| \sum_{\nu,\mu=1}^{n} \alpha'_\nu \alpha'_\mu \left[K(\eta_\nu, \eta_\mu) - \frac{f'(\eta_\nu)\overline{f'(\eta_\mu)}}{\pi[1+f(\eta_\nu)\overline{f(\eta_\mu)}]^2}\right]\right|.$$

Observe that

$$\mathrm{Re}\left\{\sum_{\nu,\mu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\Big[\chi(\eta_\nu,\eta_\mu)+u(\eta_\nu,\eta_\mu)\Big]\right\}=$$

$$=\sum_{\nu,\mu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\Big[\chi(\eta_\nu,\eta_\mu)+u(\eta_\nu,\eta_\mu)\Big].$$

Now, inequality (19) can be written in the form

$$(19')\qquad \left|\sum_{\nu,\mu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\Big[\chi_D(\eta_\nu,\eta_\mu)+u(\eta_\nu,\eta_\mu)\Big]\right|\leqslant$$

$$\leqslant\sum_{\nu,\mu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\left[K_D(\eta_\nu,\bar{\eta}_\mu)-\frac{f'(\eta_\nu)\overline{f'(\eta_\mu)}}{\pi\big[1+f(\eta_\nu)\overline{f(\eta_\mu)}\big]^2}\right].$$

Hence, we have proved the following theorem.

T h e o r e m 2. If a function f(z) conformally maps a simply - connected domain D onto the domain B described in section 1 and η_ν, $\nu=1,2,\dots,n$, are arbitrary points in D and $\zeta_\nu=f(\eta_\nu)$, then the inequality (19') holds, where α_μ are arbitrary complex numbers and the functions $\chi_D(z,\zeta)$, $K_D(z,\bar{\zeta})$ are defined by (18) and (18').

Let us put $D=K(0,r)$, $0<r<1$. Then by [2], we have

$$K_D(z,\bar{\zeta})=\frac{1}{\pi(r^2-\bar{\zeta}z)^2},\quad \chi_D(z,\zeta)=0.$$

In this case inequality (19') takes the form

$$(20)\qquad \left|\sum_{\mu,\nu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\left\{\frac{f'(\eta_\nu)f'(\eta_\mu)}{[f(\eta_\nu)-f(\eta_\mu)]^2}\right\}\right|\leqslant$$

$$\leqslant\sum_{\mu,\nu=1}^{n}\alpha_\nu\bar{\alpha}_\mu\left[\frac{1}{(r^2-\eta_\mu\eta_\nu)^2}-\frac{f'(\eta_\mu)\overline{f'(\eta_\nu)}}{\big[1+f(\eta_\mu)\overline{f(\eta_\nu)}\big]^2}\right].$$

3. Let K denote the class of functions of the form

$$f(z) = b_1 z + b_2 z^2 + \dots, \qquad b_1 > 0,$$

single-valued in the disc K(0,1) and satisfying the condition

$$f(z_1)\,\overline{f(z_2)} \neq -1 \quad \text{for} \quad z_1, z_2 \in K(0, 1) .$$

Let $B_r = f(D)$. It is easy to see that if $w \in \bar{B}_r$, then $-\frac{1}{\bar{w}} \notin \bar{B}_r$. Hence, B_r satisfies the same requirements as the domain B described in section 1. Consequently, every function $f \in K$ satisfies inequality (20). Taking limit in (20) with $r \to 1$, we obtain the following theorem.

T h e o r e m 3. If a function f belongs to the class K, then the following inequality holds

$$(21)\qquad \left| \sum_{\mu,\nu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left\{ \frac{f'(\zeta_\nu) f'(\zeta_\mu)}{\left[f(\zeta_\nu) - f(\zeta_\mu)\right]^2} \right\} \right| \leqslant$$

$$\leqslant \sum_{\nu,\mu=1}^{n} \alpha_\nu \bar{\alpha}_\mu \left[\frac{1}{(1-\zeta_\mu \bar{\zeta}_\nu)^2} - \frac{f'(\zeta_\mu)\overline{f'(\zeta_\nu)}}{\left[1+f(\zeta_\mu)\overline{f(\zeta_\nu)}\right]^2} \right].$$

In particular, if in (21) we put $\alpha_1 = \alpha_2 = 1$ and take limit with $\zeta_1 \to \zeta_2 = z$, then we obtain

$$(22)\qquad \left| \left\{ \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left[\frac{f''(z)}{f'(z)} \right]^2 \right\} \right| \leqslant \frac{1}{(1-|z|^2)^2} - \frac{|f'(z)|^2}{\left[1+|f(z)|^2\right]^2}.$$

The expression $\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left[\frac{f''(z)}{f'(z)}\right]^2$ is known as the Schwarzian of the function f. Hence, inequality (22) gives an estimation of the Schwarzian in the class K. We shall show in the sequel that this estimation is exact.

4. We now take up the problem of finding an extremal function for the inequality (22). To this aim observe that in order that a homography h(z) satisfy the condition

$$h(z_1)\,\overline{h(z_2)} \neq -1 \qquad \text{for} \quad z_1, z_2 \in K(0,1)$$

it is sufficient that

$$h(z)\, h\left(\frac{-1}{\bar z}\right) = -1 \qquad \text{for} \quad z \in K(0,1) .$$

Such a homography is the function

$$h(z) = \frac{z-a}{1+\bar a z}, \quad |a| < 1.$$

Let us form the compound function

$$f(z) = h\left(\frac{z+a}{1+\bar a z}\right) = \frac{z(1-|a|)^2}{2\bar a z + 1 + |a|^2} \tag{23}$$

The function (23) nullifies the left-hand side of (22) for every $z \in K(0,1)$. We are going to find for which z the right-hand side of (22) is 0, when we substitute (23). Since

$$\frac{|f'(z)|}{1+|f(z)|^2} = \frac{1-|a|^4}{4|a|^2|z|^2+(1+|a|^2)^2+4(1+|a|^2)\mathrm{Re}(\bar a z)+|z|^2(1+|a|^2)^2},$$

the right-hand side of (22) is 0 provided that

$$\frac{1-|a|^4}{4|a|^2|z|^2+(1+|a|^2)^2+4(1+|a|^2)\mathrm{Re}(\bar a z)+|z|^2(1-|a|^2)^2} = \tag{24}$$

$$= \frac{1}{1-|z|^2} .$$

Let us put $a = -z_0$ in (23), where $z_0 \in K(0,1)$. Then the relation (24) holds for $z = z_0$. This means that for any $z_0 \in K(0,1)$ the function (23) with $a = -z_0$ is extremal for inequality (22). In particular, if z_0 is real, the function (23) maps the circle K(0,1) onto the circle

$$\left| w - \frac{6z_0}{z_0^2-1} \right| = \frac{z_0^2+1}{1-z_0^2} .$$

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