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IDEALS AND MULTIPLIERS OF A CLASS OF INTEGRAL FUNCTIONS

The object of this paper is to study the structure of maximal ideals and principal ideals in the space of a class of integral functions $R = \left\{f \,\middle|\, f(z) = \sum_{n=0}^{\infty} a_n z^n, n!|a_n|<k, |z|<\infty\right\}$ as defined in [1] and studied in an earlier paper. Further we have studied the multiplier problem for this class.

We shall give now a characterization of maximal ideal in the space R. For this aim we introduce the following definition.

Definition. Let $A(f) \equiv [z \in C \mid f(z) = 0, f \subset R]$ and let $A(I) = [A(f) \mid f \in I \subset R]$. An ideal I of R is called <u>fixed</u> if $\bigcap_{f\in I} A(f)$ is non-empty.

Theorem 1. Every maximal fixed ideal of R is of the form

$$I(\alpha) = [f \in R \mid f(\alpha) = 0] \quad \text{for some } \alpha \in C.$$

Proof. Let for any fixed $\alpha \in C$ $I(\alpha) = [f \in R \mid f(\alpha) = 0]$. It is clear that $I(\alpha)$ is a fixed ideal of R. Now the mapping $f \to f(\alpha)$ is a homomorphism of R into C, whose kernel is $I(\alpha)$. As the kernel of a homomorphism is a maximal set, $I(\alpha)$ is a maximal fixed ideal of R. On the other hand, if I is a fixed ideal and if $\bigcap_{f\in I} A(f)$ contains more than one point, say α_1 and α_2, then I is properly contained in $I(\alpha_1)$ or $I(\alpha_2)$, violating the maximality of I.

T h e o r e m 2. If $f_1, f_2, \ldots, f_n \in R$ are such that $\hat{f}_k(\tau) \neq 0$ $(k=1,2,\ldots,n)$ for $\tau \in \Delta(R)$, where $\hat{f}_k$ is the Gelfand transform of f_k and $\Delta(R)$ is the space of maximal ideals in R, then the ideal generated by $f_1, f_2, \ldots f_n$ is a principal ideal.

We first prove the following lemma from which the proof of our theorem will follow.

L e m m a. Let A be a commutative Banach algebra with identity e. If $x_1, x_2, \ldots, x_n \in A$, then either exist some $\tau \in \Delta(A)$ such that $\hat{x}_k(\tau) = 0$ $(k=1,2,\ldots,n)$ or there exist $y_1, y_2, \ldots, y_n \in A$ such that $\sum_{k=1}^{n} x_k y_k = e$.

P r o o f. The elements of A are either singular or regular. If $x_1, x_2, \ldots, x_n$ are all singular, then $0 \in \sigma(x_k)$, $(k=1,2,\ldots,n)$. We also know that if A has an identity, then for $\alpha \in A$ $\sigma(x) = \mathcal{R}(\hat{x})$, where $\mathcal{R}(\hat{x})$ is the range of $\hat{x}$. Hence there exists some $\tau \in \Delta(A)$ such that $\hat{x}_k(\tau) = 0$ $(k=1,2,\ldots,n)$.

On the other hand, if x_k $(k=1,2,\ldots,n)$ are all regular, then for every x_k $(k=1,2,\ldots,n)$ we have one y'_k such that $x_k y'_k = e$. Hence it follows that $\sum_{k=1}^{n} x_k y_k = e$, where $y_1, y_2, \ldots, y_n \in A$.

P r o o f of T h e o r e m 2. If the condition of Theorem 2 holds, then by the lemma we have elements $g_1, g_2, \ldots \ldots, g_n \in R$ such that $f_1(z) \circ g_1(z) + f_2(z) \circ g_2(z) + \ldots + f_n(z) \circ g_n(z) = e^z$. This shows that the ideal generated by $f_1, f_2, \ldots, f_n$ is nothing but the principal ideal generated by the identity element e^z.

C o r o l l a r y. Any finite set $f_1, f_2, \ldots, f_n$ for which $\hat{f}_k(\tau) \neq 0$ $(k=1,2,\ldots,n)$, $\tau \in \Delta(R)$, generates the ring R.

P r o o f. As the ideal R is generated by the identity element e^z and the ideal generated by $f_1, f_2, \ldots, f_n$ is the principal ideal generated by the identity ideal (e^z), so $f_1, f_2, \ldots, f_n \in R$ $(k=1,2,\ldots,n)$ for which $\hat{f}_k(\tau) \neq 0$, $\tau \in A(R)$, generate the ring R. We consider now the set $W = \{f \in R \mid f(z) = 0$ at a finite set of given points$\}$. It is easy to see that W is an ideal in R. We shall prove that W is a closed set in R.

It is enough to prove that the set of functions which vanish at $z = z_0$ is closed. If $f_p \to f$, where $f_p \in W$, $(p=1,2,\ldots,n)$ is to be shown that f also vanish at $z = z_0$ Now given $\varepsilon > 0$ we can find p_0 such that $\|f_p - f\| < \varepsilon$ for $p > p_0$ i.e.

$$\sup_n n! \left|a_{pn} - a_n\right| < \varepsilon \Rightarrow \left|a_{pn} - a_n\right| < \varepsilon' \quad \text{for} \quad p > p_0 .$$

This, together with the result [2] that the convergence in norm in R is equivalent to the uniform convergence in any finite circle of the complex plane, shows that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in W$. Hence W is closed.

P r o p o s i t i o n. W is a set of first category.

P r o o f. We shall show that W is non-dense in R. If this is not true, then W contains a ball B whose centre is f_0. If $f \in R$ is such that $f(z_0) \neq 0$, then $f_0 + \lambda f$ does not vanish at $z = z_0$ for $\lambda \neq 0$. But since R is a normed linear space, we have $\|\lambda f\| \to 0$ as $|\lambda| \to 0$. Hence for sufficiently small λ, $f_0 + \lambda f \in W$ which is a contradiction. Hence W is non-dense in R.

Now we shall consider the multipliers of H^p into R, where H^p is the Hardy class of functions [4] with pth mean bounded.

T h e o r e m 3. A necessary and sufficient condition for a sequence $\{\lambda_n\}$ to be multuplier of H^p into R is that

$$\lambda_n = O\left(\frac{n^{1-1/p}}{n!}\right) \quad \text{where} \quad 0 < p \leqslant 1 .$$

P r o o f:

1° If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ and the condition $\lambda_n = O\left(\frac{n^{1-1/p}}{n!}\right)$ holds, then $|a_n| \leqslant Cn^{1/p-1}\|f\|_p$ and $|\lambda_n| < K\,\frac{n^{1-1/p}}{n!}$. This shows that $n!\,|a_n\lambda_n| < K' \Rightarrow \sum_{n=0}^{\infty} a_n\lambda_n z^n \in R$. Hence the condition is sufficient.

2° If $\{\lambda_n\}$ is a multiplier, then by the closed graph theorem, the operator Λ fixed by $\{\lambda_n\}$ is a bounded operator from H^p into R i.e. $\sup_n n!|\lambda_n a_n| \leqslant C\,\|f\|_p$, $f \in H^p$.

If we take $f(z) = g(\gamma z)$, where $\gamma < 1$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $b_n \sim K\,n^{1/p}$, then we have

$$n!\,|\lambda_n|\; n^{1/p}.\gamma^n \quad C(1-\gamma)^{-1}.$$

Putting $\gamma = 1-1/n$ we get

$$|\lambda_n| = O\left(\frac{n^{1-1/p}}{n!}\right),$$

which proves necessity of the condition.

BIBLIOGRAPHY

[1] M.K. S e n: On a class of integral functions, Bull, Cal. Math. Soc. 61 (1969) 67-74.

[2] P.K. D a s: Space of a class of integral functions. (in print).

[3] R. L a r s e n: Banach algebra. New York 1973.

[4] P.L. D u r e n: Theory of H^p spaces. New York 1970.

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