DEMONSTRATIO MATHEMATICA
Vol. IX No 2 1976

R. Bittner, E. Sadowska

ELEMENTS OF FINITE VARIATION

Suppose we are given two linear lattices L^o, L^1 ($L^1 \subset L^o$) and two linear operations: a derivative $S: L^1 \to L^o$ (onto) and an integral $T: L^o \to L^1$ such that $ST = id_{L^o}$ (see papers [2], [3]). This paper deals with the relations between a partial-order in L^o and a partial-order in the set Incr (see Definition 2.1) of elements x, such that $Sx \geqslant 0$. There are considered (o)-convergence and regular (r)-convergence of x_n in Var (see Section 4), which are determined by such convergences of Sx_n.

1. Introduction

Let X be a linear space.

Definition 1.1. A set $W \subset X$ is said to be a wedge, when its elements satisfy the following conditions

(1) $$\text{if } x,y \in W, \text{ then } x + y \in W,$$

(2) $$\text{if } x \in W, \ \lambda \geqslant 0, \text{ then } \lambda x \in W.$$

A wedge W will be called a cone, when its elements satisfy moreover the condition

(3) $$\text{if } x \in W \ (x \neq 0), \text{ then } -x \notin W.$$

Let $K \subset X$ be a cone. We define in the linear space X a relation of partial-order $\leqslant$ by

(4) $$x \leqslant y \Leftrightarrow y - x \in K.$$

A cone $K \subset X$ will be called reproducal, if every element $x \in X$ can be written as the difference of elements of the cone K.

D e f i n i t i o n 1.2. The linear partially ordered space X, such that every two-element subset of the space has a supremum and an infimum, will be called linear lattice.

Let X be a linear lattice. For every element $x \in X$ we introduce

$$(5) \qquad x_+ = \sup(x,0), \quad x_- = \sup(-x,0), \quad x = \sup(x,-x).$$

D e f i n i t i o n 1.3. An operation $A: X \longrightarrow X$ will be called positive, if $AK \subset K$, where $K \subset X$ is a cone.

Let L^0 be a linear lattice in which the relation of partial order is induced by a cone $K \subset L^0$. Suppose we are given a linear lattice L^0 and a sublattice L^1 and two linear operations $S: L^1 \rightarrow L^0$ (onto $L^1 \subset L^0$) and $T: L^0 \longrightarrow L^1$ (into) such that $Stf = f$ for $f \in L^0$. The operation S will be called a derivative, the operation T will be called an integral.

The elements $c \in \operatorname{Ker} S$ such that $Sc = 0$ will be called constant. The operation s from L^1 into the set of constant ($sx = x - TSx$, $x \in L^1$) is linear (see [2], [3]). A partial-order in L^1 is induced by the partial-order in L^0.

2. The properties of the operations m and μ

D e f i n i t i o n 2.1. Let Incr denote the set of values of the operation m

$$(6) \qquad m(x) = x + T(Sx)_- \qquad \text{for } x \in L^1,$$

and let Decr denote the set of values of the operation μ.

$$(7) \qquad \mu(x) = x - T(Sx)_+ \qquad \text{for } x \in L^1.$$

The elements of the set Incr (Decr) will be called increasing (decreasing) elements.

D e f i n i t i o n 2.2

$$(8) \qquad \operatorname{Ker}_{\geqslant} S = \left\{ x \in L^1 : Sx \geqslant 0 \right\},$$

$$(9) \qquad \operatorname{Ker}_{\leqslant} S = \left\{ x \in L^1 : Sx \leqslant 0 \right\}.$$

T h e o r e m 2.1.

(10) $$\mathrm{Incr} = \mathrm{Ker}_{\geqslant} S$$

(11) $$\mathrm{Decr} = \mathrm{Ker}_{\leqslant} S.$$

P r o o f.

1° Let $m_1 = m(x) = x + T(Sx)_-$. We get

$$Sm(x) = Sx + ST(Sx)_- = (Sx)_+ - (Sx)_- + (Sx)_-$$

that is $Sm(x) = (Sx)_+$, but $(Sx)_+ \geqslant 0$, hence $m_1 \in \mathrm{Ker}_{\geqslant} S$ or $\mathrm{Incr} \subset \mathrm{Ker}_{\geqslant} S$.

2° Let $m_1 \in \mathrm{Ker}_{\geqslant} S$. We have $Sm_1 \geqslant 0$, hence $(Sm_1)_- = 0$, but $m(m_1) = m_1 + T(Sm_1)_- = m_1$, hence $\mathrm{Ker}_{\geqslant} S \subset \mathrm{Incr}$.

The proof of (11) is analogous.

The operations m and μ have certain dual properties so that the proofs of the theorems for operations m and μ are similar.

T h e o r e m 2.2. If the integral T is a positive operation, then the operation m is convex and the operation μ is concave.

P r o o f. We will show that for $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 \geqslant 0$ we have

(12) $$m(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 m(x_1) + \alpha_2 m(x_2).$$

Since $(x+y)_- \leqslant x_- + y_-$ (see [7]), we have

$$m(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 x_1 + \alpha_2 x_2 + T\left[S(\alpha_1 x_1 + \alpha_2 x_2)\right]_- \leqslant$$

$$\leqslant \alpha_1 x_1 + \alpha_2 x_2 + T(\alpha_1 Sx_1)_- + T(\alpha_2 Sx_2)_- =$$

$$= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 T(Sx_1)_- + \alpha_2 T(Sx_2)_-,$$

hence we get (12). From definition (7) we have

$$\mu(\alpha_1 x_1 + \alpha_2 x_2) \geqslant \alpha_1 \mu(x_1) + \alpha_2 \mu(x_2)$$

for $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 \geqslant 0$.

T h e o r e m 2.3. The operations m and μ have for $x \in L^1$ the following properties

(13) $$m(m(x)) = m(x),$$

(14) $$\mu(\mu(x)) = \mu(x),$$

(15) $$m(\mu(x)) = \mu(m(x)) = sx .$$

If the integral T is positive, then

(16) $$m(x+y) \leqslant m(x) + m(y) ,$$

(17) $$\mu(x+y) \geqslant \mu(x) + \mu(y) .$$

If $\alpha > 0$, then

(18) $$m(\alpha x) = \alpha m(x) ,$$

(19) $$\mu(\alpha x) = \alpha \mu(x) .$$

If $\alpha < 0$, then

(20) $$m(\alpha x) = \alpha \mu(x) ,$$

(21) $$\mu(\alpha x) = \alpha m(x) .$$

Moreover, we have

(22) $$m(0) = 0,$$

(23) $$\mu(0) = 0,$$

(24) $$m(-x) = -\mu(x) ,$$

(25) $$\mu(-x) = -m(x),$$

(26) $$m(x) + m(-x) = T|Sx| ,$$

(27) $$\mu(x) + \mu(-x) = -T|Sx|.$$

The proofs can easily be obtained from definitions (6) and (7) and from the properties of elements of the linear lattice L^1.

T h e o r e m 2.4. Any element $x \in L^1$ can be represented in the form

(28) $$x = -x_0 + m(x) - m(-x) ,$$

where $x_0 \in \operatorname{Ker} S$ and $m(x), m(-x) \in \operatorname{Incr}$.

P r o o f. If $x \in L^1$, then $Sx \in L^0$. We have

$$Sx = (Sx)_+ - (Sx)_- \quad \text{and} \quad TSx = T(Sx)_+ - T(Sx)_-.$$

From definitions (6) and (7) we have

(29) $$x = -x_0 + \mu(x) + m(x).$$

From (24) we obtain (28).

3. <u>The properties of the Incr. Quasi-order in L^1</u>

It is easy to prove the following theorems.

T h e o r e m 3.1.

(30) If $x \in \text{Incr}$, then $m(x) = x$.

T h e o r e m 3.2. A set Incr is a wedge.

D e f i n i t i o n 3.1. Let $x_1, x_2 \in L^1$. We define in L^1 a relation $\prec$ in tne following manner

(31) $$x_1 \prec x_2 \Leftrightarrow x_2 - x_1 \in \text{Incr i.e. } S(x_2 - x_1) \geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow x_2 = x_1 + m_1, \; m_1 \in \text{Incr}.$$

T h e o r e m 3.3. The relation $\prec$ defined by formula (31) is reflexive and transitive i.e. is a quasi-order.

P r o o f.

1° $x_1 \prec x_1 \Leftrightarrow S(x_1 - x_1) = 0$, $0 \in \text{Incr}$.

2° If $x_1 \prec x_2$ and $x_2 \prec x_3$, then $S(x_2 - x_1) \geqslant 0$ and $S(x_3 - x_2) \geqslant 0$. But

$$S(x_2 - x_1 + x_2 - x_2) = S(x_2 - x_1) + S(x_3 - x_2) \geqslant 0,$$

hence $S(x_3 - x_1) \geqslant 0$ i.e. $x_1 \prec x_3$.

The space L^1 is quasi-ordered by the set Incr.

4. <u>The properties of the set of classes Var</u>

Let $x, y \in L^1$. We say that

(32) $$\bar{x} \in [x] \Leftrightarrow \bar{x} = x + c, \quad \text{where} \quad c \in \text{KerS}.$$

We define

(33) $$[x] + [y] = [x+y],$$

(34) $$\lambda[x] = [\lambda x], \quad \lambda\text{-scalar}.$$

So we have

(35) $$c \in [0] \Longleftrightarrow c \in \operatorname{Ker} S.$$

The set of classes $[x]$, $x \in L^1$, will be denoted by Var and its elements will be called elements of finite variation. The set Var is a linear space.

We define an operation

(36) $$S_m : \mathrm{Var} \longrightarrow L^0$$

in the following manner

(37) $$S_m[x] = Sx, \ x \in L^1.$$

Let us introduce a set K_m in the space Var defined as follows

(38) $$K_m = \{[x] : x \in \mathrm{Incr}\}.$$

T h e o r e m 4.1. The set K_m is a cone in the space Var.

P r o o f.

If $[m_1] \in K_m$ and $[m_2] \in K_m$, then $[m_1] + [m_2] \in K_m$, because $m_1 + m_2 \in \mathrm{Incr}$. Similarly if $[m_1] \in K_m$ and $\lambda \geq 0$, then $\lambda[m_1] = [\lambda m_1] \in K_m$. If $[m_1] \in K_m$ $(m_1 \neq c)$, then $Sm_1 \geq 0$. If $-[m_1] \in K_m$, then $-Sm_1 \geq 0$ i.e. $Sm_1 \leq 0$. Hence $Sm_1 = 0$ i.e. $m_1 = c$.

Given a cone K_m we define in the space Var a partial-order $\underset{m}{\prec}$ by

(39) $$[x] \underset{m}{\prec} [y] \Leftrightarrow [y] - [x] \in K_m.$$

Hence

(40) $$[x] \underset{m}{\prec} [y] \Longleftrightarrow x \prec y.$$

D e f i n i t i o n 4.1. A linear lattice with the partial-order $\leqslant$ is called an archimedean lattice, if

$$a \geqslant 0 \quad \text{and} \quad na \leqslant b \quad \text{for } i = 1,2,\ldots \text{ implies } a = 0.$$

Let L^o be an archimedean lattice.

T h e o r e m 4.2. The cone K_m is an archimedean cone in Var.

P r o o f. We would like to show that if

(41) $$[m_1] \in K_m \quad \text{and} \quad [m_2] - n[m_1] \in K_m \quad (i = 1,2,\ldots),$$

then

(42) $$[m_1] = [0].$$

From (41) we have $Sm_1 \geqslant 0$ and $Sm_2 - nSm_1 \geqslant 0$. Since L^o is an archimedean lattice we have

$$Sm_1 = 0 \quad \text{i.e.} \quad m_1 = c \quad \text{i.e.} \quad [m_1] = [0].$$

In Theorem 2.4 there was a decomposition

(43) $$x = -x_o + m(x) + \mu(x) = -x_o + m(x) - m(-x)$$

i.e.

(44) $$[x] = [m(x)] - [m(-x)],$$

where $[m(x)], [m(-x)] \in K_m$.

Hence we have the following theorem.

T h e o r e m 4.3. Every element of the space Var can be written as the difference of elements of the K_m cone i.e. the cone K_m is reproducal.

T h e o r e m 4.4.

(45) $$[z] = \sup_{\underset{m}{\leqslant}} ([x], [x])$$

iff

(46) $$S_m[z] = \sup \left(S_m[x], \ S_m[y]\right).$$

From the definition (37) we can write (46) as follows

(47) $$Sz = \sup(Sx, Sy).$$

P r o o f. From definition (39) and from (40) we have

$$\left.\begin{array}{ll} x \prec z, & y \prec z \\ x \prec u, & y \prec u \end{array}\right\} \Longrightarrow z \prec u$$

i.e.

(48) $$\left.\begin{array}{ll} Sz \geqslant Sx, & Sz \geqslant Sy \\ Su \geqslant Sx, & Su \geqslant Sy \end{array}\right\} \Longrightarrow Su \geqslant Sz \qquad \text{i.e. (47)}$$

From (47) and from the definition of supremum in L^0 we have

$$\left.\begin{array}{ll} Sx \leqslant Sz, & Sy \leqslant Sz \\ Sx \leqslant Su, & Sy \leqslant Su \end{array}\right\} \Longrightarrow Sz \leqslant Su$$

and

$$\left.\begin{array}{ll} S(z-x) \geqslant 0, & S(z-y) \geqslant 0 \\ S(u-x) \geqslant 0, & S(u-y) \geqslant 0 \end{array}\right\} \Longrightarrow S(u-z) \geqslant 0$$

i.e.

(49) $$\left.\begin{array}{ll} x \prec z, & y \prec z \\ x \prec u, & y \prec u \end{array}\right\} \Longrightarrow z \prec u.$$

From the definition of the relation $\prec$ we obtain (45).

T h e o r e m 4.5.

(50) $$[w] = \inf_{\substack{\prec \\ m}} \big([x], [y]\big)$$

iff

(51) $$S_m[w] = \inf\big(S_m[x], S_m[y]\big).$$

From the definition (37) of S_m we can write (51) as follows

(52) $$Sw = \sup(Sx, Sy).$$

The proof is analogous to the proof of Theorem 4.4.

The linear space Var with supremum and infimum is a linear lattice.

For each element $[x] \in$ Var we define

$$(53) \qquad [x]_+ = \sup([x], [0]),$$

$$(54) \qquad [x]_- = \sup(-[x], [0]),$$

$$(55) \qquad |[x]| = \sup(-[x], [x])$$

and we call them the positive part, the negative part and the module of the element $[x]$, respectively.

From Theorem 4.4. we have the following corollary.

C o r o l l a r y.

$$(56) \qquad \sup([x], [0]) = [x]_+ = [m(x)],$$

$$(57) \qquad \sup(-[x], [0]) = [x]_- = [m(-x)],$$

$$(58) \qquad \sup(-[x], [x]) = |[x]| = [m(x) + m(-x)].$$

P r o o f. From Th. 4.4. we have

$$(59) \qquad Sm(x) = S(x+T(Sx)_-) = (Sx)_+ ,$$

because $\sup(Sx,0) = (Sx)_+$ in L^0. Similarly we obtain (57) and (58).

A linear lattice is called a K-space, if every non-empty subset is bounded from above and has a supremum. In the K-space we can introduce a convergence based on partial-order which we call (o)-convergence.

Let L^1 and L^0 be a K-space. Let x_n (n=1,2,...) be a sequence of elements from L^1. If the sequence x_n is increasing i.e. $x_n \leq x_{n+1}$, then $\lim_{n\to\infty} x_n = \sup_n x_n$. By the superior limit and the inferior limit of the sequence x_n we mean the elements

$$(60) \qquad \overline{\lim_{n\to\infty}}\, x_n = \inf_n \left[\sup(x_n, x_{n+1}, \dots)\right],$$

$$(61) \qquad \underline{\lim_{n\to\infty}}\, x_n = \sup_n \left[\inf(x_n, x_{n+1}, \dots)\right].$$

We say that a sequence x_n is (o)-convergent to an element x, if

$$\overline{\lim_{n\to\infty}}\ x_n = \underline{\lim_{n\to\infty}}\ x_n = x . \tag{62}$$

The element x will be called a (o)-limit and we write $(o)\text{-}\lim\limits_{n\to\infty} x_n = x$. If the sequence x_n is increasing, then $\lim\limits_{n\to\infty} [x_n] \overset{df}{=} \left[\lim\limits_{n\to\infty} x_n\right]$. Let a sequence x_n be bounded. Then in the space Var there exist supremum and infimum

$$\left[\sup(x_n, x_{n+1}, \dots)\right] \overset{df}{=} \lim_k \sup_k(\left[x_n\right], \left[x_{n+1}\right], \dots) , \tag{63}$$

$$\left[\inf(x_n, x_{n+1}, \dots)\right] \overset{df}{=} \lim_k \inf_k(\left[x_n\right], \left[x_{n+1}\right], \dots). \tag{64}$$

By definition we have

$$\overline{\lim_{n\to\infty}}\ \left[x_n\right] = \left[\inf \sup(x_n, x_{n+1}, \dots)\right], \tag{65}$$

$$\underline{\lim_{n\to\infty}}\ \left[x_n\right] = \left[\sup \inf(x_n, x_{n+1}, \dots)\right]. \tag{66}$$

We have the following general definition of a (o)-convergence in the K-space.

D e f i n i t i o n 4.2. A sequence $[x_n]$ is (o)-convergent to a class $[x]$, if

$$\overline{\lim_{n\to\infty}}\ \left[x_n\right] = \underline{\lim_{n\to\infty}}\ \left[x_n\right]. \tag{67}$$

The common value of limit superior and limit inferior will be called (o)-limit and we write

$$(o)\text{-}\lim_{n\to\infty} \left[x_n\right] = \left[x\right]. \tag{68}$$

T h e o r e m 4.6.

$$(o)\text{-}\lim_{n\to\infty} \left[x_n\right] = \left[x\right], \tag{69}$$

iff

(70) $$\overline{\lim_{n\to\infty}}\, Sx_n = \underline{\lim_{n\to\infty}}\, Sx_n$$

or

$$(o)\text{-}\lim_{n\to\infty} Sx_n = Sx .$$

P r o o f.

1^{o} If $(o)\text{-}\lim_{n\to\infty} [x_n] = [x]$, then

(71) $$\overline{\lim_{n\to\infty}}\, [x_n] = \underline{\lim_{n\to\infty}}\, [x_n] = [x].$$

Let

(72) $$\overline{\lim_{n\to\infty}}\, [x_n] = [G] \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, [x_n] = [g].$$

From (46), (47), by passing to limit, we obtain

(73) $$S_m[G] = \overline{\lim_{n\to\infty}}\; S_m[x_n] = [\inf \sup(Sx_n, Sx_{n+1}, \dots,)],$$

(74) $$S_m[g] = \underline{\lim_{n\to\infty}}\; S_m[x_n] = [\sup \inf(Sx_n, Sx_{n+1}, \dots)].$$

The equalities (73) and (74) are equivalent to the following equalities

(75) $$SG = \overline{\lim_{n\to\infty}}\; Sx_n ,$$

(76) $$Sg = \underline{\lim_{n\to\infty}}\; Sx_n .$$

From (72) we have

$$[G] = [g] \text{ i.e. } G = g + c \quad \text{and} \quad SG = Sg ,$$

hence

$$\overline{\lim_{n\to\infty}}\; Sx_n = \underline{\lim_{n\to\infty}}\; Sx_n .$$

2^{o} If $(o)\text{-}\lim_{n\to\infty} Sx_n = Sx$ or $SG = Sg$ i.e. $S(G-g) = 0$, then

$$G = g + c \quad \text{and} \quad [G] = [g] .$$

From (65) we have

$$\overline{\lim_{n\to\infty}}\,[x_n] = \underline{\lim_{n\to\infty}}\,[x_n].$$

We have proved in Th. 4.2 that a cone K_m in the space Var is an archimedean cone.

So, by the general definition, we can introduce in this archimedean lattice the notion of a regular convergence.

D e f i n i t i o n 4.4. The sequence $[x_n]$ is (r)-convergent to an element $[x]$, if

$$(77) \qquad \bigvee [f] \underset{m}{\succ} [0], \quad \bigwedge_{\varepsilon > 0} \ \bigvee_{N} \ \bigwedge_{n > N} \left|[x_n] - [x]\right| \underset{m}{\prec} \varepsilon [f]$$

and we then write

$$(78) \qquad (r)\text{-}\lim_{n\to\infty} [x_n] = [x].$$

T h e o r e m 4.7.

$$(79) \qquad (r)\text{-}\lim_{n\to\infty} [x_n] = [x],$$

iff

$$(80) \qquad (r)\text{-}\lim_{n\to\infty} Sx_n = Sx .$$

P r o o f.

1° If $(r)\text{-}\lim_{n\to\infty} [x_n] = [x]$, then (77),

i.e.

$$-\varepsilon[f] \underset{m}{\prec} [x_n] - [x] \underset{m}{\prec} \varepsilon[f].$$

From the definition (39) of the relation $\underset{m}{\prec}$ we have

$$-\varepsilon f \prec x_n - x \prec \varepsilon f$$

i.e.

$$-\varepsilon Sf \leqslant S(x_n - x) \leqslant \varepsilon Sf ,$$

hence

$$|Sx_n - Sx| \leqslant \varepsilon Sf.$$

2^0 If $(r)\text{-}\lim_{n\to\infty} Sx_n = Sx$, then $|Sx_n - Sx| \leqslant \varepsilon Sf$

i.e.

$$[-\varepsilon Sf] \leqslant [Sx_n - Sx] \leqslant [\varepsilon Sf],$$

hence

(81) $$[-\varepsilon Sf] + [Sm_1] = [Sx_n - Sx]$$

and

(82) $$[Sx_n - Sx] = [\varepsilon Sf] - [Sm_2].$$

From (81) we obtain

$$S_m([-\varepsilon f] + [m_1] - [x_n - x]) = [0]$$

or $[x_n - x] = [-\varepsilon f + m_1]$, where $m_1, m_2 \in$ Inor i.e.

$$x_n - x \succ -\varepsilon f$$

and

$$[-\varepsilon f] \underset{m}{\prec} [x_n - x].$$

Similarly from (82) we obtain

$$[x_n - x] \underset{m}{\prec} [\varepsilon f],$$

hence

$$|[x_n] - [x]| \underset{m}{\prec} \varepsilon [f].$$

This ends the proof of Theorem 4.7.

REFERENCES

[1] S. B e l l e r t: On foundations of operational calculus, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 5 (1957) 855-858.

[2] R. B i t t n e r: Operational calculus in linear spaces, Studia Math. 20 (1961) 1-18.

[3] R. B i t t n e r: Algebraic and analytic properties of solutions of abstract differential equations. Warszawa 1964.

[4] К.М. Г л е з м а н, Ю.К. Л ю б и ч: Конечномерный линейный анализ в задачах. Москва 1969.

[5] W. R u d i n: Principles of mathematical analysis. New York-San Francisco-Toronto-London 1964.

[6] Л.А. С к о р н я к о в: Элементы теории структур. Москва 1970.

[7] Б.З. В у л и х: Введение в теорию полуупорядоченных пространств. Москва 1961.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDAŃSK, 80-952 GDAŃSK-OLIWA

Received March 19, 1975.