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Sabina Kwiecień

REDUCTION OF THE EQUATION $F(x_1, x_2, x_3, x_4)=0$ TO THE FORM $f_1(x_1)f_2(x_2)+g_1(x_1)f_3(x_3)+h_1(x_1)+f_4(x_4)=0$

1. Nomogram for the equation $f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4) = 0$.

Let

(1) $$f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) = \alpha,$$

then the considered equation has the form

(2) $$\alpha + h_1(x_1) + f_4(x_4) = 0.$$

We may write equations (1) and (2) in the form

(1') $$\begin{vmatrix} 0 & \frac{\alpha}{k_1} & 1 \\ 1 & f_3(x_3) & 1 \\ \frac{g_1(x_1)}{g_1(x_1)-k_1} & \frac{f_1(x_1)f_2(x_2)}{k_1 - g_1(x_1)} & 1 \end{vmatrix} = 0,$$

(2') $$\begin{vmatrix} 0 & \frac{\alpha}{k_1} & 1 \\ -k_2 & \frac{f_4(x_4)}{k_1} & 1 \\ -\frac{1}{2}k_2 & -\frac{1}{2k_1}h_1(x_1) & 1 \end{vmatrix} = 0,$$

where $k_2 > 0$ and $0 \neq k_1 \neq g(x_1)$ in the interval in which x_1 varies. Hence we obtain that for equation (1) there exists a collineation nomogram (cf. [2]) with two scales on parallel straight lines (the scale for α is regular) and with two functional scales (x_1, x_2) (the curves x_1 are straight lines) (cf. [4]). Further, for equation (2) there exists a collineation nomogram with three scales on parallel straight lines (the scale for α is regular). Now we can infer that for equation

$$(3) \qquad f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4) = 0$$

we can construct a nomogram with uniform dummy axis α. Because of the simplicity of construction of this nomogram, considerations announced in the title of the present paper are of the great interest.

2. The necessary and sufficient condition for $G(x_1,x_2,x_3,x_4)$ to be of the form $G \equiv f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4)$, (cf. [4]).

We consider the function $G : D \to R^1$, where

$$D = \{(x_1,x_2,x_3,x_4) : a_i \leqslant x_i \leqslant b_i, \quad i = 1,2,3,4\}$$

and $G \in C^3(D)$.

T h e o r e m 1. The function G is of the form

$$G \equiv f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4)$$

if and only if the following conditions are satisfied in the set D:

$$(a) \qquad \frac{\partial^2 G}{\partial x_4\, \partial x_i} \equiv 0 \qquad \text{for } i = 1,2,3,$$

$$(b) \qquad \frac{\partial^2 G}{\partial x_2\, \partial x_3} \equiv 0,$$

$$(c) \qquad \frac{\partial^2 G}{\partial x_1\, \partial x_k} \not\equiv 0 \qquad \text{for } k = 2,3,$$

(d) $$\frac{\partial^2}{\partial x_1 \partial x_k} \ln\left|\frac{\partial G}{\partial x_k}\right| \equiv 0 \qquad \text{for } k = 2,3.$$

P r o o f. The necessity of conditions (a), (b), (c) and (d) is evident. Now we prove their sufficiency. From (a) and (b) it follows that there exist functions $A^2(x_1,x_2)$, $A^3(x_1,x_3)$ and $B_4(x_4) \not\equiv$ const such that

(4) $$G \equiv A^2(x_1,x_2) + A^3(x_1,x_3) + B_4(x_4).$$

Hence

(5) $$\frac{\partial G}{\partial x_k} = \frac{\partial A^k(x_1,x_k)}{\partial x_k} \qquad \text{for } k = 2,3.$$

From (5) and (d) we obtain

$$\frac{\partial^2}{\partial x_1 \partial x_k} \ln\left|\frac{\partial A^k(x_1,x_k)}{\partial x_k}\right| \equiv 0 .$$

Hence we infer that there exist functions $\alpha_1^k(x_1)$, $\alpha_k^k(x_k)$ $(k = 2,3)$ such that

$$\ln\left|\frac{\partial A^k(x_1,x_k)}{\partial x_k}\right| \equiv \alpha_1^k(x_1) + \alpha_k^k(x_k)$$

or

$$\frac{\partial A^k(x_1,x_k)}{\partial x_k} \equiv \exp\left[\alpha_1^k(x_1)\right]\cdot\exp\left[\alpha_k^k(x_k)\right].$$

Hence

(6) $$A^k(x_1,x_k) \equiv B_1^k(x_1)\cdot B_k^k(x_k) + C_1^k(x_1), \quad (k = 2,3),$$

where

$$B_1^k(x_1) = \exp\left[\alpha_1^k(x_1)\right],$$

$$B_k^k(x_k) = \int_{x_k^o}^{x_k} \exp\left[\alpha_k^k(x_k)\right] dx_k, \quad x_k^o \in \langle a_k, b_k \rangle$$

and $C_1^k(x_1)$ are arbitrary functions of x_1. In virtue of (4) and (6) we have

(7) $$G \equiv B_1^2(x_1)B_2^2(x_2) + B_1^3(x_1)B_3^3(x_3) + C_1^2(x_1) + C_1^3(x_1) + B_4(x_4).$$

We put

$$B_1^2(x_1) = f_1(x_1), \qquad B_1^3(x_1) = g_1(x_1),$$

$$C_1^2(x_1) + C_1^3(x_1) = h_1(x_1), \quad B_k^k(x_k) = f_k(x_k) \text{ for } k = 2,3,$$

$$B_4(x_4) = f_4(x_4).$$

Then we obtain

(8) $$G \equiv f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4).$$

In the last equality, in virtue of (c), the functions $f_1(x_1)$, $f_2(x_2)$, $g_1(x_1)$, $f_3(x_3)$ are non-constant, q.e.d.

3. Reduction of $F(x_1,x_2,x_3,x_4) = 0$ to the form (3).

Let the function $F : D \to R^1$ of the class $C^3(D)$ satisfy in D the condition

$$\prod_{i=1}^{4} \frac{\partial F}{\partial x_i} \neq 0.$$

T h e o r e m 2. The conditions

(A) $$\frac{F''_{x_i x_4}}{F'_{x_i} F'_{x_4}} \equiv \frac{F''_{x_2 x_3}}{F'_{x_2} F'_{x_3}}, \qquad (i = 1,2,3),$$

(B) $$\frac{\partial}{\partial x_1}\left(\frac{F'_{x_k}}{F'_{x_4}}\right) \not\equiv 0, \qquad (k = 2,3),$$

(C) $$\frac{\partial^2}{\partial x_1 \partial x_k} \ln\left|\frac{F'_{x_2}}{F'_{x_3}}\right| \equiv 0, \qquad (k = 2,3),$$

(D) $$\frac{\partial^2}{\partial x_2 \partial x_3} \ln\left|\frac{F'_{x_2}}{F'_{x_3}}\right| \equiv 0$$

(in the set D) are necessary and sufficient for the existence of anamorphosis (cf. [5]) $\phi : R^1 \to R^1$, $\phi \in C^3(R^1)$, $\phi'(u) \neq 0$, such that $\phi(F) \equiv G$.

P r o o f of sufficiency. Let

(9) $$H(x_1,x_2,x_3,x_4) \equiv \frac{F''_{x_2x_3}}{F'_{x_2}\, F'_{x_3}}.$$

We shall prove that there exists a function $\Psi(u)$ such that

(10) $$H(x_1,x_2,x_3,x_4) \equiv \Psi(F).$$

From (9) we obtain

(11) $$\frac{H'_{x_i}}{F'_{x_i}} \equiv \frac{1}{F'_{x_2}F'_{x_3}F'_{x_i}}\left(F'''_{x_2x_3x_i} - \frac{F''_{x_2x_3}F''_{x_2x_i}}{F_{x_2}} - \frac{F''_{x_2x_3}F''_{x_3x_i}}{F_{x_3}}\right),$$

for $i = 1,2,3,4$. Hence for $i = 3$ we have

(12) $$\frac{H'_{x_3}}{F'_{x_3}} \equiv \frac{1}{F'_{x_2}(F'_{x_3})^2}\left[F'''_{x_2x_3x_3} - \frac{(F''_{x_2x_3})^2}{F'_{x_2}} - \frac{F''_{x_2x_3}F''_{x_3x_3}}{F'_{x_3}}\right].$$

Using condition (D) which takes the form

$$\frac{\partial}{\partial x_2}\left(\frac{F''_{x_2x_3}}{F'_{x_2}} - \frac{F''_{x_3x_3}}{F'_{x_3}}\right) \equiv 0$$

or

$$\frac{F'''_{x_2x_2x_3}}{F'_{x_2}} - \frac{F''_{x_2x_2}F''_{x_2x_3}}{(F'_{x_2})^2} \equiv \frac{F'''_{x_2x_3x_3}}{F'_{x_3}} - \frac{F''_{x_3x_3}F''_{x_2x_3}}{(F'_{x_3})^2}$$

we obtain

$$(13)\quad \frac{H'_{x_3}}{F'_{x_3}} \equiv \frac{1}{F'_{x_2}(F'_{x_3})^2}\left[F'_{x_3}\left(\frac{F'''_{x_2x_2x_3}}{F'_{x_2}} - \frac{F''_{x_2x_3}F''_{x_2x_2}}{(F'_{x_2})^2}\right) + \right.$$

$$\left. - \frac{(F''_{x_2x_3})^2}{F'_{x_2}}\right] \equiv \frac{1}{(F'_{x_2})^2F'_{x_3}}\left[F'''_{x_2x_2x_3} - \frac{F''_{x_2x_2}F''_{x_2x_3}}{F'_{x_2}} + \right.$$

$$\left. - \frac{(F''_{x_2x_3})^2}{F'_{x_3}}\right] \equiv \frac{H'_{x_2}}{F'_{x_2}} .$$

The last identity follows from identity (11) for $i = 2$. By condition (A) and equality (9) we have

$$H \equiv \frac{F''_{x_ix_4}}{F'_{x_i}F'_{x_4}} \qquad \text{for } i = 1,2,3.$$

Therefore

$$(14)\quad \frac{H'_{x_k}}{F'_{x_k}} \equiv \frac{1}{F'_{x_i}F'_{x_k}F'_{x_4}}\left(F'''_{x_ix_kx_4} - \frac{F''_{x_ix_4}F''_{x_ix_k}}{F'_{x_k}} - \frac{F''_{x_ix_4}F''_{x_kx_4}}{F'_{x_4}}\right)$$

for $i = 1,2,3$ and $k = 1,2,3,4$. Putting $i = 2$, $k = 1$ we get

$$\frac{H'_{x_1}}{F'_{x_1}} \equiv \frac{1}{F'_{x_2}F'_{x_1}F'_{x_4}}\left(F'''_{x_2x_1x_4} - \frac{F''_{x_2x_4}F''_{x_2x_1}}{F'_{x_2}} - \frac{F''_{x_2x_4}F''_{x_1x_4}}{F'_{x_4}}\right) \equiv$$

$$\equiv \frac{1}{F'_{x_1}F'_{x_2}F'_{x_4}}\left(F'''_{x_1x_2x_4} - \frac{F''_{x_2x_4}F''_{x_1x_2}}{F'_{x_2}} - \frac{F''_{x_1x_4}F''_{x_2x_4}}{F'_{x_4}}\right);$$

by (A) we have

$$\frac{F''_{x_1x_4}}{F'_{x_1}} \equiv \frac{F''_{x_2x_4}}{F'_{x_2}}.$$

Hence

$$\frac{H'_{x_1}}{F'_{x_1}} \equiv \frac{1}{F'_{x_1}F'_{x_2}F'_{x_4}}\left(F'''_{x_1x_2x_4} - \frac{F''_{x_1x_4}F''_{x_1x_2}}{F'_{x_1}} - \frac{F''_{x_1x_4}F''_{x_2x_4}}{F'_{x_4}}\right) \equiv \frac{H'_{x_2}}{F'_{x_2}} \tag{15}$$

which follows from (14) for $i = 1$, $k = 2$. In virtue of identity (11) for $i = 4$ we have

$$\frac{H'_{x_4}}{F'_{x_4}} \equiv \frac{1}{F'_{x_2}F'_{x_3}F'_{x_4}}\left(F'''_{x_2x_3x_4} - \frac{F''_{x_2x_3}F''_{x_2x_4}}{F'_{x_2}} - \frac{F''_{x_2x_3}F''_{x_3x_4}}{F'_{x_3}}\right).$$

In virtue of condition (A) we get

$$\frac{F''_{x_2x_3}}{F'_{x_3}} \equiv \frac{F''_{x_2x_4}}{F'_{x_4}}.$$

Hence and from (14) we get

$$(16)\quad \frac{H'_{x_4}}{F'_{x_4}} \equiv \frac{1}{F'_{x_2}F'_{x_3}F'_{x_4}}\left(F'''_{x_2x_3x_4} - \frac{F''_{x_2x_3}F''_{x_2x_4}}{F'_{x_2}} - \frac{F''_{x_2x_4}F''_{x_3x_4}}{F'_{x_4}}\right) \equiv \frac{H'_{x_3}}{F'_{x_3}}$$

Using (13), (15) and (16) we obtain

$$\frac{H'_{x_1}}{F'_{x_1}} \equiv \frac{H'_{x_2}}{F'_{x_2}} \equiv \frac{H'_{x_3}}{F'_{x_3}} \equiv \frac{H'_{x_4}}{F'_{x_4}}.$$

Thus we can infer that there exists a function $\Psi(u)$ (cf. [1]) such that

$$H(x_1,x_2,x_3,x_4) \equiv \Psi(F).$$

Let the function $\phi(F)$ be a solution of the differential equation

$$(17)\qquad \frac{y''}{y'} = -\Psi(F).$$

Then we have the identity

$$\frac{\phi''(F)}{\phi'(F)} + \Psi(F) \equiv 0.$$

We shall prove that the function $\phi(F)$ defined above is the required anamorphosis. Let $\phi(F) \equiv G(x_1,x_2,x_3,x_4)$. The function G satisfies the condition of Theorem 1. The condition (a) is satisfied, because

$$\frac{\partial^2 G}{\partial x_4 \partial x_i} \equiv \frac{\partial}{\partial x_4}(\phi' F'_{x_i}) \equiv \phi'' F'_{x_4}F'_{x_i} + \phi' F''_{x_i x_4} \equiv$$

$$\equiv \phi' F'_{x_4}F'_{x_i}\left(\frac{\phi''}{\phi'} + \frac{F''_{x_i x_4}}{F'_{x_i}F'_{x_4}}\right) \equiv \phi' F'_{x_4}F'_{x_i}\left(\frac{\phi''}{\phi'} + \Psi(F)\right) \equiv 0$$

for $i = 1,2,3$. Condition (b) is satisfied, because

$$\frac{\partial^2 G}{\partial x_2 \partial x_3} \equiv \phi'' F'_{x_2} F'_{x_3} + \phi' F''_{x_2 x_3} \equiv \phi' F'_{x_2} F'_{x_3} \left(\frac{\phi''}{\phi'} + \frac{F''_{x_2 x_3}}{F'_{x_2} F'_{x_3}} \right) \equiv$$

$$\equiv \phi' F'_{x_2} F'_{x_3} \left(\frac{\phi''}{\phi'} + \Psi(F) \right) \equiv 0.$$

Further, for $k = 2,3$, we have

$$\frac{\partial^2 G}{\partial x_1 \partial x_k} \equiv \phi'' F'_{x_1} F'_{x_k} + \phi' F''_{x_1 x_k} \equiv \phi' F'_{x_1} F'_{x_k} \left(\frac{\phi''}{\phi'} + \frac{F''_{x_1 x_k}}{F'_{x_1} F'_{x_k}} \right) \equiv$$

$$\equiv \phi' F'_{x_1} F'_{x_k} \left(\frac{F''_{x_1 x_k}}{F'_{x_1} F'_{x_k}} - \frac{F''_{x_1 x_4}}{F'_{x_1} F'_{x_4}} \right) \equiv \phi' F'_{x_4} \left(\frac{F''_{x_k x_1}}{F'_{x_4}} - \frac{F''_{x_4 x_1} F'_{x_k}}{(F'_{x_4})^2} \right) \equiv$$

$$\equiv \phi' F'_{x_4} \frac{\partial}{\partial x_1} \left(\frac{F'_{x_k}}{F'_{x_4}} \right) \not\equiv 0$$

(in virtue of assumption (B)), which proves condition (c). Finally we have

$$\frac{\partial^2}{\partial x_1 \partial x_k} \ln \left| G'_{x_k} \right| \equiv \frac{\partial^2}{\partial x_1 \partial x_k} \left(\ln |\phi'| + \ln \left| F'_{x_k} \right| \right) \equiv$$

$$\equiv \frac{\partial}{\partial x_1} \left(\frac{\phi'' F'_{x_k}}{\phi'} + \frac{F''_{x_k x_k}}{F'_{x_k}} \right) \equiv \frac{\partial}{\partial x_1} \left(\frac{F''_{x_k x_k}}{F'_{x_k}} - \frac{F''_{x_2 x_3}}{F'_{x_2} F'_{x_3}} F'_{x_k} \right), \quad (k = 2,3).$$

Hence, for $k = 2$, we obtain

$$\frac{\partial^2}{\partial x_1 \partial x_2} \ln \left| G'_{x_2} \right| \equiv \frac{\partial}{\partial x_1} \left(\frac{F''_{x_2 x_2}}{F'_{x_2}} - \frac{F''_{x_2 x_3}}{F'_{x_3}} \right) \equiv \frac{\partial^2}{\partial x_1 \partial x_2} \ln \left| \frac{F'_{x_2}}{F'_{x_3}} \right|$$

and for $k = 3$

$$\frac{\partial^2}{\partial x_1 \partial x_3} \ln \left| G'_{x_3} \right| \equiv \frac{\partial}{\partial x_1} \left(\frac{F''_{x_3x_3}}{F'_{x_3}} - \frac{F''_{x_2x_3}}{F_{x_2}} \right) \equiv - \frac{\partial^2}{\partial x_1 \partial x_3} \ln \left| \frac{F'_{x_2}}{F'_{x_3}} \right| .$$

Therefore, in virtue of condition (C), we have

$$\frac{\partial^2}{\partial x_1 \partial x_k} \ln \left| G'_{x_k} \right| \equiv 0 \quad \text{for} \quad k = 2,3,$$

i.e. condition (d) is satisfied. According to Theorem 1 there exist functions $f_1(x_1)$, $g_1(x_1)$, $h_1(x_1)$, $f_2(x_2)$, $f_3(x_3)$, $f_4(x_4)$ such that $G \equiv \phi(F) \equiv f_1f_2 + g_1f_3 + h_1 + f_4$.

Thus the function $\phi(F)$ being a solution of equation (17) is a required anamorphosis, this completes the proof of sufficiency of conditions in Theorem 2.

Necessity of these conditions may be proved by simple operations with the function $F(x_1,x_2,x_3,x_4) \equiv \equiv \phi^{-1}(f_1f_2 + g_1f_3 + h_1 + f_4)$, where ϕ^{-1} is a function inverse to the anamorphosis ϕ. The existence of ϕ^{-1} is assured by definition of ϕ as a solution of equation (17).

T h e o r e m 3. Anamorphosis $\varphi: R^1 \to R^1$, $\varphi \in C^3(R^1)$, $\varphi'(u) \neq 0$ reducing the equation $F(x_1,x_2,x_3,x_4) = 0$ to the form (3) exists if and only if the conditions (A), (B), (C) and (D) of Theorem 2 are satisfied in the set D.

P r o o f. It is easy to show that the required anamorphosis is the function $\varphi(u) = \phi(u) - \phi(0)$, where ϕ satisfies equation (17). It follows from the fact that φ is an anamorphosis reducing function F to the form

$$f_1(x_1)f_2(x_2) + g_1(x_1)f_3(x_3) + h_1(x_1) + f_4(x_4)$$

and $\varphi(u) = 0$ if and only if $u = 0$ (by definition of φ and in virtue of the condition $\varphi'(u) \neq 0$), q.e.d.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

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