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PROPERTIES OF A CERTAIN SINGULAR INTEGRAL AND THEIR APPLICATIONS TO SOLUTION OF HILBERT PROBLEM IN THE SPACE

1. Introduction

Let S be a closed Ljapunov surface which is the boundary of a domain D^+ in the space E_3. The complement of the set $D^+ \cup S$ (in E_3) we denote by D^-.

We shall use the following notation (cf. [1], p. 168):

$$D(X,Y,Z) = \begin{Vmatrix} 0 & X & Y & Z \\ X & 0 & -Z & Y \\ Y & Z & 0 & -X \\ Z & -Y & X & 0 \end{Vmatrix}, \quad D^*(X,Y,Z) = \begin{Vmatrix} 0 & X & Y & Z \\ X & 0 & Z & -Y \\ Y & -Z & 0 & X \\ Z & Y & -X & 0 \end{Vmatrix},$$

and

$$(1) \qquad M(A,Q) = -D^*\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\frac{1}{|A-Q|}\cdot D(\alpha,\beta,\gamma),$$

where $|A - Q|$ is the Euclidean distance of $A(x,y,z) \in D^+ \cup D^-$, $Q(\xi,\eta,\zeta) \in S$ and $N(\alpha,\beta,\gamma)$ is a unit vector orthogonal to the surface S at the point Q, directed outside the domain D^+.

Let the elements $q_i(Q)$, $i = 1,2,3,4$, of the column-vector

$$(2) \qquad q(Q) = \begin{Vmatrix} q_1(Q) \\ q_2(Q) \\ q_3(Q) \\ q_4(Q) \end{Vmatrix}$$

be defined and continuous on S. The surface integral

$$\frac{1}{4\pi}\int_S M(A,Q)q(Q)dS_Q ,$$

(Cauchy integral) is defined for every $A \in D^+ \cup D^-$.

It is known [1] that

1° The column-vector

$$F(A) = \frac{1}{4\pi}\int_S M(A,Q)q(Q)dS_Q \tag{3}$$

is holomorphic in $D^+ \cup D^-$, i.e. it satisfies in this set the elliptic system of equations

$$D\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F(A) = 0.$$

2° If the elements of (2) satisfy the Hölder condition on S, then for every $P \in S$ there exists a singular integral (Cauchy principal value)

$$\frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q \tag{4}$$

defined by the equality

$$\frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q = \frac{1}{4\pi}\int_S M(P,Q)\left[q(Q)-q(P)\right]dS_Q + \frac{1}{2}\, q(P); \tag{5}$$

besides, the formulae (corresponding to Plemelj formulae)

$$\begin{aligned} F^+(P) &= \frac{1}{2}\, q(P) + \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q , \\ F^-(P) &= -\frac{1}{2}\, q(P) + \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q \end{aligned} \tag{6}$$

hold, where $F^+(P)$ and $F^-(P)$ are boundary values of the column-vector $F(A)$ defined by the equalities

$$F^+(P) = \lim_{\substack{A\to P\in S\\ (A\in D^+)}} F(A), \qquad F^-(P) = \lim_{\substack{A\to P\in S\\ (A\in D^-)}} F(A).$$

Moreover (cf. [2], p. 43) the following theorem of Żakowski holds.

If the elements $q_i(Q)$, $i = 1,2,3,4$, of the column-vector (2) satisfy on S the conditions

$$|q_i(Q)| \leqslant K, \quad |q_i(Q)-q_i(Q_1)| \leqslant K|Q-Q_1|^h, \quad 0<h<1,$$

where K is a positive constant, then the elements $F_i(P)$, $i = 1,2,3,4$, of the column-vector $F(P) = \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q$ satisfy the inequalities

$$|F_i(P)| \leqslant CK, \quad |F_i(P)-F_i(P_1)| \leqslant CK|P-P_1|^h,$$

for $P\in S$, $P_1\in S$, where C is a positive constant independent of $q(Q)$.

This theorem is an analogue of the theorem of Privalov in the theory of one-dimensional singular Cauchy integral.

2. Investigation of a certain singular Cauchy integral in the space E_3.

Let Σ denote the plane $z = 0$ in the space E_3 of variables x,y,z and D^+ and D^- be the upper $(z > 0)$ and lower $(z < 0)$ half-space, respectively. Suppose that the elements $g_i(Q)$, $i = 1,2,3,4$, of the column-vector $g(Q)$ are defined and continuous in the plane Σ and satisfy the inequalities

(7) $$|g_i(Q_1)-g_i(Q_2)| \leqslant C|Q_1-Q_2|^h \text{ (in every bounded subset of } \Sigma\text{)},$$
$$|g_i(Q)| \leqslant \frac{C}{|Q|^h} \text{ for } |Q| > R_0,$$

where R_0, h, C are arbitrary positive constant.

The integral

$$(8) \qquad \phi(A) = \frac{1}{4\pi}\int\limits_{\Sigma} M(A,Q)g(Q)dQ, \quad A(x,y,z) \in E_3 - \Sigma, \quad Q(\xi,\eta,\zeta) \in \Sigma$$

we define as a limit

$$(9) \qquad \frac{1}{4\pi}\int\limits_{\Sigma} M(A,Q)g(Q)dQ = \lim_{\varrho\to\infty} \frac{1}{4\pi}\int\limits_{\Omega} M(A,Q)g(Q)dQ\,,$$

where Ω is a simply connected domain in Σ containing the origin 0 of coordinates, boundary of which is a piecewise smooth curve, and ϱ is the least distance of 0 to the points of the boundary of Ω. Because the elements $M_{i,k}(A,Q)$ $(i = 1,2,3,4;\ k = 1,2,3,4)$ of the matrix $M(A,Q)$ have the estimation

$$(10) \qquad \left|M_{i,k}(A,Q)\right| \leqslant \frac{1}{|A-Q|^2},$$

the inequality (7) is sufficient for the existence of the integral (8) defined by (9).

We write the integral (9) in the following way

$$(11) \qquad \frac{1}{4\pi}\int\limits_{\Sigma} M(A,Q)g(Q)dQ =$$

$$= \lim_{\varrho\to\infty}\left[\frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)\left[g(Q)-g(P)\right]dQ + \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)g(P)dQ +\right.$$

$$\left. + \frac{1}{4\pi}\int\limits_{\Omega-K(P,R)} M(A,Q)g(Q)dQ\right] = \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)\left[g(Q)-g(P)\right]dQ +$$

$$+ \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)g(P)dQ + \frac{1}{4\pi}\int\limits_{\Sigma-K(P,R)} M(A,Q)g(Q)dQ\,,$$

where $P(x,y)$ is an arbitrary point of the domain Ω and $K(P,R)$ is a circle with centre in P and radius R such that $K(P,R) \subset \Omega$. We suppose that $\Omega \subset K(0,R_0)$. In particular, for

$$g(P) = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}$$

we have

$$\frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)g(P)dQ = \begin{Vmatrix} \frac{1}{4\pi}\int\limits_{K(P,R)} \frac{z}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)^3}\,d\xi d\eta \\ \frac{1}{4\pi}\int\limits_{K(P,R)} \frac{-(y-\eta)}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)^3}\,d\xi d\eta \\ \frac{1}{4\pi}\int\limits_{K(P,R)} \frac{x-\xi}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)^3}\,d\xi d\eta \\ 0 \end{Vmatrix}.$$

Introducing the polar coordinates by the formulae $x - \xi = r\cos\varphi$, $y - \eta = r\sin\varphi$, we obtain

(12) $$\frac{1}{4\pi}\int\limits_{K(P,R)} \frac{z}{\left(\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}\right)^3}\,d\xi d\eta =$$

$$= \frac{z}{4\pi}\int_0^{2\pi}\int_0^{R} \frac{r}{\left(\sqrt{r^2 + z^2}\right)^3}\,dr d\varphi = \frac{1}{2}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}}\right).$$

Similarly,

(13) $$\frac{1}{4\pi}\int\limits_{K(P,R)} \frac{-(y-\eta)}{\left(\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}\right)^3}\,d\xi d\eta = 0$$

and

$$(14)\qquad \frac{1}{4\pi}\int\limits_{K(P,R)} \frac{x-\xi}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)^3}\, d\xi d\eta = 0.$$

In virtue of (12), (13) and (14), we obtain

$$(15)\qquad \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)dQ = \begin{cases} \frac{1}{2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)E & \text{for } A\in D^+ \\ -\frac{1}{2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)E & \text{for } A\in D^-, \end{cases}$$

where E is the identity matrix.

Let $P(x,y)$ be the ortogonal projection of the point $A(x,y,z)$ onto the plane Σ. Then we define the integral (9) by the formula

$$(16)\qquad \frac{1}{4\pi}\int\limits_{\Sigma} M(P,Q)g(Q)dQ = \lim_{\varrho\to\infty}\lim_{\delta\to 0}\left[\frac{1}{4\pi}\int\limits_{\Omega-K(P,\delta)} M(P,Q)g(Q)dQ\right] =$$

$$= \lim_{\varrho\to\infty}\lim_{\delta\to 0}\left[\frac{1}{4\pi}\int\limits_{K(P,R)} M(P,Q)\left[g(Q)-g(P)\right]dQ + \right.$$

$$\left. + \frac{1}{4\pi}\int\limits_{K(P,R)-K(P,\delta)} M(P,Q)g(P)dQ + \frac{1}{4\pi}\int\limits_{\Omega-K(P,R)} M(P,Q)g(Q)dQ\right],$$

where $0<\delta<R$.

In particular, for $g(P) = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}$ we have

$$\frac{1}{4\pi}\int\limits_{K(P,R)-K(P,\delta)} M(P,Q)g(P)dQ = \left\| \begin{matrix} 0 \\ \dfrac{1}{4\pi}\displaystyle\int\limits_{K(P,R)-K(P,\delta)} \frac{-(y-\eta)}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^3}\, d\xi d\eta \\ \dfrac{1}{4\pi}\displaystyle\int\limits_{K(P,R)-K(P,\delta)} \frac{x-\xi}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^3}\, d\xi d\eta \\ 0 \end{matrix} \right\|$$

Analogously as in the proof of (12), we obtain

$$(17)\qquad \frac{1}{4\pi}\int\limits_{K(P,R)-K(P,\delta)} \frac{x-\xi}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^3}\, d\xi d\eta = \int_0^{2\pi}\int_\delta^R \frac{r^2\sin\varphi}{r^3}\, dr d\varphi = 0$$

and

$$(18)\qquad \frac{1}{4\pi}\int\limits_{K(P,R)-K(P,\delta)} \frac{-(y-\eta)}{\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^3}\, d\xi d\eta = 0 .$$

In virtue of (17) and (18), we may write

$$(19)\qquad \frac{1}{4\pi}\int\limits_{K(P,R)-K(P,\delta)} M(P,Q)dQ = \|0\| ,$$

where $\|0\|$ is the zero matrix.

Thus we can write the singular integral (16) in the form

$$(20)\qquad \frac{1}{4\pi}\int\limits_{\Sigma} M(P,Q)g(Q)dQ = \frac{1}{4\pi}\int\limits_{K(P,R)} M(P,Q)\left[g(Q)-g(P)\right]dQ + $$

$$+ \frac{1}{4\pi}\int\limits_{\Sigma-K(P,R)} M(P,Q)g(Q)dQ .$$

By a simple calculation we can verify that the column-vector $\phi(A)$ defined by (8) is holomorphic separately in D^+ and D^-.

Let $\phi^+(P) = \lim\limits_{\substack{A \to P \\ (A \in D^+)}} \phi(A)$, $\phi^-(P) = \lim\limits_{\substack{A \to P \\ (A \in D^-)}} \phi(A)$.

L e m m a. If elements $g_i(Q)$, $i = 1,2,3,4$, of the column-vector $g(Q)$ satisfy the Hölder condition, then for the integral (8) the Plemelj formulae are satisfied

$$
(21) \qquad \begin{cases} \phi^+(P) = \frac{1}{2}\, g(P) + \frac{1}{4\pi}\int\limits_{\Sigma} M(P,Q)g(Q)dQ\,, \\ \phi^-(P) = -\,\frac{1}{2}\, g(P) + \frac{1}{4\pi}\int\limits_{\Sigma} M(P,Q)g(Q)dQ\,. \end{cases}
$$

P r o o f. We write the integral (8) in the form (11) and, by (15), we obtain

$$
(22) \quad \phi(A) = \begin{cases} \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)\left[g(Q)-g(P)\right]dQ + \frac{1}{4\pi}\int\limits_{\Sigma - K(P,R)} M(A,Q)g(Q)dQ \, + \\ \qquad + \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) g(P), \quad A \in D^+, \\ \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)\left[g(Q)-g(P)\right]dQ + \frac{1}{4\pi}\int\limits_{\Sigma - K(P,R)} M(A,Q)g(Q)dQ \, + \\ \qquad - \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) g(P), \quad A \in D^-. \end{cases}
$$

Since

$$
\lim_{A \to P} \frac{1}{4\pi}\int\limits_{K(P,R)} M(A,Q)\left[g(Q)-g(P)\right]dQ = \frac{1}{4\pi}\int\limits_{K(P,R)} M(P,Q)\left[g(Q)-g(P)\right]dQ
$$

and

$$\lim_{A\to P} \frac{1}{4\pi} \int\limits_{\Sigma - K(P,R)} M(A,Q)g(Q)dQ = \frac{1}{4\pi} \int\limits_{\Sigma - K(P,R)} M(P,Q)g(Q)dQ\,,$$

we have, in virtue of (22),

$$(23)\quad \begin{cases} \phi^{+}(P) = \frac{1}{2}\, g(P) + \frac{1}{4\pi} \int\limits_{K(P,R)} M(P,Q)\left[g(Q)-g(P)\right]dQ \; + \\ \qquad\qquad + \frac{1}{4\pi} \int\limits_{\Sigma - K(P,R)} M(P,Q)g(Q)dQ\,, \\ \phi^{-}(P) = -\frac{1}{2}\, g(P) + \frac{1}{4\pi} \int\limits_{K(P,R)} M(P,Q)\left[g(Q)-g(P)\right]dQ \; + \\ \qquad\qquad + \frac{1}{4\pi} \int\limits_{\Sigma - K(P,R)} M(P,Q)g(Q)dQ\,. \end{cases}$$

Then, making use of (20), we obtain (21).

Remark. Formulae (21) have been proved under the supposition that the point $A(x,y,z) \in D^{+} \cup D^{-}$ tends to $P(x,y)$ in parallel to the z-axis. The truth of the Plemelj formulae, when A tends to P arbitrarily, follows from the fact that $\phi(A)$ uniformly tends to $\phi^{\pm}(P)$ along the parallels to the z-axis and from the continuity of boundary values $\phi^{\pm}(P)$.

3. The Hilbert problem

Problem. Find a vector function $\phi(A)$ holomorphic in D^{+} and in D^{-} separately, bounded in whole space E_3, such that the boundary values $\phi^{+}(P)$ and $\phi^{-}(P)$ satisfy in every point $P \in \Sigma$ the condition

$$(24)\qquad \phi^{+}(P) = G\cdot\phi^{-}(P) + g(P).$$

We suppose that G is a matrix with constant elements of the form

$$G = \begin{Vmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -\beta_2 & \beta_1 & -\beta_4 & \beta_3 \\ -\beta_3 & \beta_4 & \beta_1 & -\beta_2 \\ -\beta_4 & -\beta_3 & \beta_2 & \beta_1 \end{Vmatrix}$$

and the elements $g_i(P)$, $i = 1,2,3,4$, of the column-vector $g(P)$ satisfy conditions (7).

In virtue of the Plemelj formulae (21), we infer that the solution of the problem (24) is the column-vector

$$(25) \qquad \phi(A) = \frac{X(A)}{4\pi} \int_{\Sigma} M(A,Q)(X^{+})^{-1} g(Q)dQ + X(A)C,$$

where

$$X(A) = \begin{cases} X^{+}(A) = G, & \text{for } A \in D^{+} \\ X^{-}(A) = E, & \text{for } A \in D^{-} \end{cases}$$

is a solution of corresponding homogeneous problem

$$X^{+}(P) = GX^{-}(P), \quad P \in \Sigma,$$

C is an arbitrary constant column-vector and E is the identity matrix.

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