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STABILITY OF THE FOUR-POINT FINITE DIFFERENCE SCHEME FOR LINEAR PARABOLIC EQUATIONS WITH FIRST DERIVATIVES AND WITH VARIABLE COEFFICIENTS

This paper presents a sufficient condition of stability of the four-point finite difference scheme for the equation

$$(1)\quad \frac{\partial u}{\partial t} = \sum_{p=1}^{n} a_p(x_1,\dots,x_n,t)\,\frac{\partial^2 u}{\partial x_p^2} + \sum_{p=1}^{n} b_p(x_1,\dots,x_n,t)\,\frac{\partial u}{\partial x_p} +$$

$$+ c(x_1,\dots,x_n,t)u + d(x_1,\dots,x_n,t)$$

with initial condition

$$(2)\quad u(x_1,\dots,x_n,0) = f_1(x_1,\dots,x_n) \quad \text{for} \quad (x_1,\dots,x_n)\in\omega_x$$

and boundary condition

$$(3)\quad u(x_1,\dots,x_n,t) =$$

$$= f_2(x_1,\dots,x_n,t) \quad \text{for} \quad (x_1,\dots,x_n)\in\Gamma, \quad t\in\omega_t .$$

The cube ω_x is given by inequalities

$$\xi_p \leqslant x_p \leqslant \zeta_p, \quad p = 1,2,\dots,n, \quad (\xi_p,\ \zeta_p \ \text{being constant}),$$

Γ is the boundary of ω_x and $\omega_t = \langle 0,T\rangle$ (T being a positive number).

Assume that

1° the coefficients are continuous on the set $\omega = \omega_x \times \omega_t$, and that

(4) $a_p(x_1,\dots,x_n,t) > 0$, for $p = 1,2,\dots,n$, on ω

(5) $$c(x_1,\dots,x_n,t) < 0 \quad \text{on } \omega.$$

2^o The functions f_1 and f_2 are continuous on ω_x and $\Gamma \times \omega_t$ resp.

Let us introduce in the set ω a rectangular grid consisting of lines intersecting each x_p axis at points distant by a grid-step $l_p = \alpha_p h$, $p = 1,2,\dots,n$, and denote by k the step on the time axis t, where $l_p = \frac{\mathfrak{Z}_p - \xi_p}{m_p}$ and numbers m_p are integer. For convenience let us define a constant

(6) $$\sigma = \frac{k}{h^2}.$$

The mesh-points of this grid are given as

$$X_{\underline{i}s} = (x_{10} + i_1 l_1,\ x_{20} + i_2 l_2,\dots,x_{n0} + i_n l_n, sk) =$$
$$= (x_{1i_1}, x_{2i_2},\dots,x_{ni_n}, t_s),$$

where the elements of $\underline{i} = [i_1,i_2,\dots,i_p,\dots,i_n]$ are integers defined by $i_p \in [0, m_p]$ and the integer $s \in [0, m_s]$, where $m_s \cdot k = T$. The following notation seems very convenient

$$a_p(X_{\underline{i}s}) = a^p_{i_1,\dots,i_n,s},\ b_p(X_{\underline{i}s}) = b^p_{i_1,\dots,i_n,s},\ (p = 1,\dots,n),$$

$$c(X_{\underline{i}s}) = c_{i_1,\dots,i_n,s},\ d(X_{\underline{i}s}) = d_{i_1,\dots,i_n,s},\ u(X_{\underline{i}s}) = u_{i_1,\dots,i_n,s}.$$

The approximations of the derivatives are obvious

(7) $$\frac{\partial u(X_{\underline{i}s})}{\partial x_p} = \frac{u_{i_1,\dots,i_p+1,\dots,i_n,s} - u_{i_1,\dots,i_p-1,\dots,i_n,s}}{2\, l_p} + R_{1p}h^2,$$

(8) $$\frac{\partial^2 u(X_{\underline{i}s})}{\partial x_p^2} =$$

$$= \frac{u_{i_1,\dots,i_p+1,\dots i_n,s} - 2u_{i_1,\dots,i_p,\dots,i_n,s} + u_{i_1,\dots,i_p-1,\dots,i_n,s}}{l_p^2} + R_{2p}h^2$$

$$(9)\qquad \frac{\partial u(X_{is})}{\partial t} = \frac{u_{i_1,\dots,i_n,s+1} - u_{i_1,\dots,i_n s}}{k} + R_3 k.$$

The values R_1, R_2, R_3 are the rest-terms.

We substitute (7), (8) and (9) into the equation (1), we neglect the rest-terms and we obtain the finite difference equation. This finite difference equation has the form

$$(10)\quad u_{i_1,\dots,i_n,s+1} = 1-\sigma\left(\sum_{p=1}^{n} 2\frac{1}{\alpha_p^2}\, a^p_{i_1,\dots,i_n,s} - h^2 c_{i_1,\dots,i_n,s}\right) +$$

$$+\sigma\sum_{p=1}^{n} u_{i_1,\dots,i_p+1,\dots,i_n,s}\left(\frac{1}{\alpha_p^2}\, a^p_{i_1,\dots,i_n,s} + \frac{h}{2\alpha_p}\, b^p_{i_1,\dots,i_n,s}\right) +$$

$$+\sigma\sum_{p=1}^{n} u_{i_1,\dots,i_p-1,\dots,i_n,s}\left(\frac{1}{\alpha_p^2}\, a^p_{i_1,\dots,i_n,s} - \frac{h}{2\alpha_p}\, b^p_{i_1,\dots,i_n,s}\right) +$$

$$+\ \sigma h^2\, d_{i_1,\dots,i_n,s}\,.$$

Denote by $u_{i_1,\dots,i_n,s}$ a solution of the problem (10), (2), (3) and by $v_{i_1,\dots,i_n,s}$ a solution of the problem (10), (2'), (3) where (2') is the following initial condition

$$(2')\qquad v(x_1,\dots,x_n,0) = f_1(x_1,\dots,x_n) + \varphi(x_1,\dots,x_n) \quad \text{for} \quad (x_1,\dots,x_n)\in\omega_x,$$

where the function φ is continuous. The difference

$$(11)\qquad \varepsilon_{i_1,\dots,i_n,s} = v_{i_1,\dots,i_n,s} - u_{i_1,\dots,i_n,s}$$

satisfies the equation (10) in which we neglect the expression $h^2 d_{i_1,\dots,i_n,s}$. If the point X_{is} belongs to the boundary Γ, then $\varepsilon_{i_1,\dots,i_n,s} = 0$. The initial condition has the form

$$\varepsilon_{i_1,\dots,i_n,0} = \varphi(x_{1i_1},\dots,x_{ni_n}).$$

It is easy to see that the expression $\varepsilon_{i_1,\ldots,i_n,s}$ is the error of the solution of the problem (10), (2), (3). The solution is stable if always is fulfiled the condition

(12) $$\left|\varepsilon_{i_1,\ldots,i_n,s+1}\right| \leqslant \left|\varepsilon_{i_1,\ldots,i_n,s}\right|.$$

Denote $\varepsilon^s = \max_i \left|\varepsilon_{i_1,\ldots,i_n,s}\right|$. Then the following estimate holds

(13) $$\left|\varepsilon_{i_1,\ldots,i_n,s}\right| \leqslant \varepsilon^s \, E_{i_1,\ldots,i_n,s} ,$$

where

(14) $$E_{i_1,\ldots,i_n,s} = \left|1-\sigma\left(2\sum_{p=1}^{n}\frac{1}{\alpha_p^2}\, a^p_{i_1,\ldots,i_n,s} - h^2 c_{i_1,\ldots,i_n,s}\right)\right| + $$

$$+\,\sigma\sum_{p=1}^{n}\left|\frac{1}{\alpha_p^2}\, a^p_{i_1,\ldots,i_n,s} + \frac{h}{2\alpha_p}\, b^p_{i_1,\ldots,i_n,s}\right| +$$

$$+\,\sigma\sum_{p=1}^{n}\left|\frac{1}{\alpha_p^2}\, a^p_{i_1,\ldots,i_n,s} - \frac{h}{2\alpha_p}\, b^p_{i_1,\ldots,i_n,s}\right| .$$

The above definition of ε^s implies that there exists such a vector of indexes $[i_1,\ldots,i_n]$ that

$$\varepsilon^{s+1} = \left|\varepsilon_{i_1,\ldots,i_n,s+1}\right| .$$

If $E^s = \max_{\underline{i}} E_{i_1,\ldots,i_n,s}$, then

(15) $$\varepsilon^{s+1} \leqslant \varepsilon^s E^s \quad \text{for all integer } s.$$

It is easy to see that if the following inequalities hold

(16) $$1 \geqslant 1 - \sigma\left(\sum_{p=1}^{n}\frac{2}{\alpha_p^2}\, a^p_{i_1,\ldots,i_n,s} - h^2 c_{i_1,\ldots,i_n,s}\right) \geqslant 0 ,$$

(17) $$\frac{1}{\alpha_p^2} a^p_{i_1,\dots,i_n,s} \pm \frac{h}{2\alpha_p} b^p_{i_1,\dots,i_n,s} \geq 0, \quad p = 1,2,\dots,n,$$

then the following condition is satisfied

(18) $$E^s = 1 + h^2 c_{i_1,\dots,i_n,s} \leq 1.$$

Hence the condition of stability is

(19) $$\varepsilon^{s+1} \leq \varepsilon^s \quad \text{for all integer} \quad s.$$

Introduce the symbols $B_p = \max\left|b_p(x_1,\dots,x_n,t)\right|$, $\bar{a}_p = \min\left|a_p(x_1,\dots,x_n,t)\right|$.

It is always possible to choose a sufficiently small grid-step l_p to fulfil the inequalities

(20) $$\bar{a}_p - \frac{1}{2} \cdot B_p l_p \geq 0, \quad p = 1,2,\dots,n,$$

which result from (17). Hence the condition for the length of grid-step is

(21) $$l_p \leq \frac{2\bar{a}_p}{B_p},$$

of course this condition can be used, when $B_p \neq 0$.
If $B_p = 0$, then $b_p(x_1,\dots,x_n,t) \equiv 0$, and the step l_p can be chosen arbitrarily. If $h = \min\limits_{p=1,\dots,n} l_p = l_r$, then $\alpha_p = \frac{l_p}{l_r}$.

From the inequality (16), we can immediately obtain the condition

(22) $$\sigma \leq \frac{1}{\sum\limits_{p=1}^{n} \frac{1}{\alpha_p^2} A_p + h^2 C},$$

where $A_p = \max\limits_{\omega} a_p(x_1,\dots,x_n,t)$, $C = \max\limits_{\omega}\left[-c(x_1,\dots,x_n,t)\right]$.

The above considerations complete the proof of the following theorem.

T h e o r e m. If the coefficients a_p, b_p, d and c of the equation (1) are continuous and fulfil the conditions (4), (5) and if the assumptions 1^o and 2^o are fulfiled and the steps l_p satisfy the condition (21), and the coefficient σ satisfies the condition (22), then the solution of the problem (10), (2), (3) is stable.

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