DEMONSTRATIO MATHEMATICA
Vol. IX No 1 1976

Józef Tabor

ON THE TRANSLATION EQUATION OCCURRING IN THE ITERATION THEORY

Iterations of invertible modulus or submodulus functions for real exponents are defined by the translation equation and some initial condition (cf. [1], p. 197; [2], p.272). Satisfying the translation equation is understood in the iteration theory in a different way than usually. In this paper we investigate the translation equation in the sense of the iteration theory. In particular, the problem of extending this equation is being studied.

We shall denote by $F : X \to Y$ any function defined in X and taking its values in Y, by $F : X \multimap Y$ any function defined in any subset of X and taking its values in Y. For a given function $F : X \multimap Y$ the symbol $x \longmapsto F(x)$ will denote the function whose value at x equals F(x).

Let E be any set and $f : E \to E$ any invertible function. Iterations of the function f may be defined for real exponents in the following way.

D e f i n i t i o n 1 (cf. [2], p.272). A one-parameter family of functions $x \to F(x,\alpha)$ defined in some subset of E is called a set of iterations of the function f, whenever the function $F: E \times (-\infty,\infty) \multimap E$ satisfies the following conditions:
(a) $F(x,\alpha)$ is defined for arbitrary $x \in E$ and arbitrary $\alpha \geq 0$,
(b) If $\alpha < 0$ and $x \in F(E, -\alpha)$, then $F(x,\alpha)$ is defined,

(c) If $F(F(x,\alpha),\beta)$ and $F(x,\alpha+\beta)$ are defined, then

$$F(F(x,\alpha),\beta) = F(x,\alpha+\beta),$$

(d) $F(x,1) = f(x)$ for $x \in E$.

It has been proved in [2] p. 276 that condition (b') in Definition 1 may be replaced by the following

(b) $F(x,\alpha)$ is defined for $\alpha < 0$ iff $x \in F(E,-\alpha)$.

We shall prove a few theorems related to a form of functions F satisfying conditions (a), (b), (c). At first, we shall quote two lemmas from note [2].

Lemma 1 (cf. [2], p.272). If $F : E \times \langle 0, \,) \to E$ satisfies the functional equation

(1) $$F(F(x,\alpha),\beta) = F(x,\alpha+\beta) \quad \text{for} \quad x \in E, \quad \alpha,\beta \in \langle 0,\infty)$$

and for some $\alpha_0 > 0$ the function $x \mapsto F(x,\alpha_0)$ is invertible, then

$$F(x,0) = x \quad \text{for} \quad x \in E.$$

Lemma 2 (cf. [2] p.276). If $F : E \times \langle 0,\infty) \to E$ satisfies functional equation (1) and for some $\alpha_0 > 0$ the function $x \mapsto F(x,\alpha_0)$ is invertible, then the function F can be extended to a function $\bar{F} : E \times (-\infty,\infty) \to E$ satisfying conditions (a), (b), (c). This extension is unique and for $\alpha < 0$ it is defined by the relation

(2) $$\bar{F}(x,\alpha) = y \quad \text{iff} \quad F(y,-\alpha) = x.$$

Let $F : E \times \langle 0,\infty) \to E$ be an arbitrary function. Let us consider the following conditions on F:

(A) The function F can be extended to a function $\tilde{F} : E \times (-\infty,\infty) \to E$ satisfying conditions (a), (b), (c).

(B) For $x_1, x_2 \in E$, $\alpha \in \langle 0,\infty)$ if $F(x_1,\alpha) = F(x_2,\alpha)$, then $F(x_1,0) = F(x_2,0)$.

(C) The function F can be represented in the form

(3) $$F(x,\alpha) = F^*(f_0(x),\alpha) \quad \text{for} \quad x \in E, \quad \alpha \in \langle 0,\infty),$$

where the function $f_0 : E \to E$ satisfies the functional equation

(4) $$f_0\big(f_0(x)\big) = f_0(x) \quad \text{for} \quad x \in E,$$

the function $F^* : f_0(E) \times \langle 0,\infty) \to f_0(E)$ satisfies functional equation (1) and for arbitrary $\alpha \in \langle 0,\infty)$ the function $f_0(E) \ni x \longmapsto F^*(x,\alpha)$ is invertible.

It is easy to prove (cf. also [1], p.306 Theorem 15.14) that the general solution of functional equation (4) can be written in the form

(5) $$f_0(x) = \begin{cases} x & \text{for} \quad x \in E_0 \\ g(x) & \text{for} \quad x \in E \setminus E_0 \end{cases}$$

where $\phi \neq E_0 \subset E$ and $g : E \setminus E_0 \to E_0$ is an arbitrary function.

If $f_0 : E \to E$ satisfies (4), then we obtain from (5) for $x \in E$

(6) $$x \in f_0(E) \quad \text{iff} \quad f_0(x) = x.$$

We shall prove the following theorem.

T h e o r e m 1. If $F = E \times \langle 0,\infty) \to E$ satisfies the functional equation (1), then conditions (A), (B), (C) are equivalent.

P r o o f . (A) $\Rightarrow$ (B). Let $\tilde{F} : E \times (-\infty,\infty) \multimap E$ be an extension of F and let $\tilde{F}$ satisfy conditions (a), (b), (c). For $x \in E$ and $\alpha \geqslant 0$ we have: $\tilde{F}(x,\alpha) \in \tilde{F}(E,\alpha)$. Consequently, it follows from (b) that $\tilde{F}(\tilde{F}(x,\alpha),-\alpha)$ is defined. Let $F(x_1,\alpha) = F(x_2,\alpha)$. Then $\tilde{F}(x_1,\alpha) = \tilde{F}(x_2,\alpha)$. Since $\tilde{F}(\tilde{F}(x_1,\alpha),-\alpha)$ and $\tilde{F}(\tilde{F}(x_2,\alpha),-\alpha)$ are defined, we have $F(x_1,0) = \tilde{F}(x_1,0) = \tilde{F}(\tilde{F}(x_1,\alpha),-\alpha) = \tilde{F}(\tilde{F}(x_2,\alpha),-\alpha) = \tilde{F}(x_2,0) = F(x_2,0)$.

(B) $\Rightarrow$ (C). Let us assume that $F : E \times \langle 0,\infty) \to E$ satisfies equation (1) and condition (B). We put

(7) $$f_0(x) := F(x,0) \quad \text{for} \quad x \in E,$$

(8) $F^*(x,\alpha) := F(x,\alpha)$ for $x \in f_0(E)$, $\alpha \in \langle 0,\infty)$.

We obtain from (1)

$$f_0(f_0(x)) = F(F(x,0),0) = F(x,0) = f_0(x).$$

Thus f_0 satisfies (4). For arbitrary $x \in E$, $\alpha \in \langle 0,\infty)$ we have

$$F(x,\alpha) = F(F(x,\alpha),0) \in f_0(E),$$

and hence for arbitrary $x \in f_0(E)$, $\alpha \in \langle 0,\infty)$, $F^*(x,\alpha)$ belongs to $f_0(E)$. For arbitrary $x \in f_0(E)$, $\alpha,\beta \in \langle 0,\infty)$ we have now

$$F^*(F^*(x,\alpha),\beta) = F(F(x,\alpha),\beta) = F(x,\alpha+\beta) = F^*(x,\alpha+\beta).$$

We have shown that F^* satisfies (1) on the set $f_0(E) \times \langle 0,\infty)$. We are going to prove that the function $f_0(E) \ni x \longmapsto F^*(x,\alpha)$ is invertible. Let $x_1, x_2 \in f_0(E)$ and $F^*(x_1,\alpha) = F^*(x_2,\alpha)$. Then by (8) $F(x_1,\alpha) = F(x_2,\alpha)$ and, in consequence of (B), $F(x_1,0) = F(x_2,0)$. Since $x_1, x_2 \in f_0(E)$, by applying (6) and (7) it follows that $x_1 = x_2$. This proves that the function $f_0(E) \ni x \longmapsto F^*(x,\alpha)$ is invertible. It follows from (7), (8) and (1) that

$$F(x,\alpha) = F(F(x,0),\alpha) = F^*(f_0(x),\alpha) \quad \text{for} \quad x \in E, \ \alpha \in \langle 0,\infty).$$

Thus, the function F can be written in form (3).

(C) $\Rightarrow$ (A). Let F be of the form (3). The function F^* can be extended by applying Lemma 2 to a function $\bar{F} : f_0(E) \times (-\infty,\infty) \to f_0(E)$ satisfying conditions (a), (b), (c). We put

$$(9) \qquad \tilde{F}(x,\alpha) := \begin{cases} F(x,\alpha) & \text{for} \quad \alpha \geq 0, \quad x \in E \\ \bar{F}(x,\alpha) & \text{for} \quad \alpha < 0, \quad (x,\alpha) \in D_{\bar{F}}^{\,*)} \end{cases}$$

*) $D_{\bar{F}}$ denotes the domain of the function $\bar{F}$.

It is obvious that $\tilde{F}$ satisfies condition (a). We shall show that $\tilde{F}$ satisfies condition (b). Since $\bar{F}$ satisfies condition (b) we have for $\alpha < 0$:

(10) $\bar{F}(x,\alpha)$ is defined iff $x \in \bar{F}(f_0(E), -\alpha)$.

If $\alpha < 0$, then $-\alpha > 0$ and $F^*(x,-\alpha) = \bar{F}(x, -\alpha)$ for $x \in f_0(E)$. Consequently, in virtue of (3), (9), we obtain

(11) $\bar{F}(f_0(E),-\alpha) = F^*(f_0(E),-\alpha) = F(E,-\alpha) = \tilde{F}(E,-\alpha)$.

It follows from (9), (10), (11) that $\tilde{F}$ satisfies condition (b). We must verify yet that $\tilde{F}$ satisfies condition (c). We shall consider for this purpose the following cases:

(i) $\alpha \geq 0$, $\beta \geq 0$,
(ii) $\alpha < 0$, $\beta < 0$,
(iii) $\alpha \geq 0$, $\beta < 0$, $\alpha+\beta \geq 0$,
(iv) $\alpha \geq 0$, $\beta < 0$ $\alpha+\beta < 0$,
(v) $\alpha < 0$, $\beta \geq 0$, $\alpha+\beta \geq 0$,
(vi) $\alpha < 0$, $\beta \geq 0$, $\alpha+\beta < 0$.

The verification of condition (c) in the cases (i) and (ii) is trivial (as F and $\bar{F}$ satisfy (c)).

Case (iii). Let $\tilde{F}(\tilde{F}(x,\alpha),\beta)$ be defined. By the definition of $\bar{F}$ we have

(12) $\bar{F}(x,\alpha) = F^*(x,\alpha)$ for $x \in f_0(E)$, $\alpha \geq 0$.

In virtue of (3), (9) and (12) we obtain in the case under consideration

$$\tilde{F}(\tilde{F}(x,\alpha),\beta) = \bar{F}(F(x,\alpha),\beta) = \bar{F}(F^*(f_0(x),\alpha),\beta) =$$

$$= \bar{F}(\bar{F}(f_0(x),\alpha),\beta) = \bar{F}(f_0(x),\alpha+\beta) = F^*(f_0(x),\alpha+\beta) =$$

$$= F(x,\alpha+\beta) = \tilde{F}(x,\alpha+\beta).$$

Case (vi). Let $\tilde{F}(\tilde{F}(x,\alpha),\beta)$ and $\tilde{F}(x,\alpha+\beta)$ be defined, i.e. let $F(\bar{F}(x,\alpha),\beta)$ and $\bar{F}(x,\alpha+\beta)$ be defined.

Since the values of the function $\bar{F}$ belong to $f_0(E)$, we have in virtue of (6)

$$(13) \qquad f_0(\bar{F}(x,\alpha)) = \bar{F}(x,\alpha).$$

Since $\bar{F}$ satisfies condition (c) and $\bar{F}(x,\alpha+\beta)$ is defined, we obtain by applying (3), (9), (12) and (13)

$$\tilde{F}(\tilde{F}(x,\alpha),\beta) = F(\bar{F}(x,\alpha),\beta) = F^*(f_0(\bar{F}(x,\alpha)),\beta) =$$
$$= F^*(\bar{F}(x,\alpha),\beta) = \bar{F}(\bar{F}(x,\alpha),\beta) = \bar{F}(x,\alpha+\beta) = \tilde{F}(x,\alpha+\beta).$$

Thus, in this case $\tilde{F}$ satisfies condition (c). The proof of condition (c) in the cases (iv) and (v) is similar to the proof in the cases (iii) and (vi). This statement completes the proof of Theorem 1.

R e m a r k 1. In virtue of (3) and (12) we have for $x \in E, \alpha \geqslant 0$:

$$F(x,\alpha) = F^*(f_0(x),\alpha) = \bar{F}(\bar{f}_0(x),\alpha).$$

Thus, formula (9) may be written in an equivalent form as follows

$$(14) \qquad \tilde{F}(x,\alpha) = \begin{cases} \bar{F}(f_0(x),\alpha) & \text{for } \alpha \geqslant 0, \quad x \in E \\ \bar{F}(x,\alpha) & \text{for } \alpha < 0, \ (x,\alpha) \in D_{\bar{F}}. \end{cases}$$

T h e o r e m 2. If $F : E \times \langle 0,\infty) \to E$ can be extended to a function $\tilde{F} : E \times (-\infty,\infty) \multimap E$ satisfying conditions (a), (b), (c), then this extension is unique, and hence it is of form (9).

P r o o f . Let $\tilde{F} : E \times (-\infty,\infty) \multimap E$ and $\tilde{\tilde{F}} : E \times (-\infty,\infty) \multimap E$ be two extensions of F and let $\tilde{F}$ and $\tilde{\tilde{F}}$ satisfy conditions (a), (b), (c). We have

$$(15) \qquad \tilde{F}(x,\alpha) = F(x,\alpha) = \tilde{\tilde{F}}(x,\alpha) \quad \text{for} \quad x \in E, \ \alpha \in \langle 0,\infty).$$

Since $\tilde{F}$ and $\tilde{\tilde{F}}$ satisfy (b), we obtain from (15) for $\alpha < 0$

$$\tilde{F}(x,\alpha) \quad \text{is defined iff} \quad x \in \tilde{F}(E,-\alpha) = F(E,-\alpha),$$

$$\tilde{\tilde{F}}(x,\alpha) \quad \text{is defined iff} \quad x \in \tilde{\tilde{F}}(E,-\alpha) = F(E,-\alpha).$$

Thus the domains of functions $\tilde{F}$ and $\tilde{\tilde{F}}$ are identical. Let us consider $x \in E$, $\alpha < 0$ such that $\tilde{F}(x,\alpha)$ and $\tilde{\tilde{F}}(x,\alpha)$ are defined. Then we have

$$(16) \qquad F\big(\tilde{F}(x,\alpha), -\alpha\big) = \tilde{F}\big(\tilde{F}(x,\alpha),-\alpha\big) = \tilde{F}(x,0) = F(x,0) =$$
$$= \tilde{\tilde{F}}\big(\tilde{\tilde{F}}(x,\alpha),-\alpha\big) = F\big(\tilde{\tilde{F}}(x,\alpha), -\alpha\big).$$

Since the extensions $\tilde{F}$ and $\tilde{\tilde{F}}$ satisfy condition (a), (c), F satisfies equation (1).

In virtue of Theorem 1, F satisfies condition (B). Hence, we obtain from (16)

$$(17) \qquad F\big(\tilde{F}(x,\alpha),0\big) = F\big(\tilde{\tilde{F}}(x,\alpha),0\big),$$

and consequently using (15) and (17) we have (for $\alpha < 0$, $(x,\alpha) \in D_{\tilde{F}}$)

$$\tilde{F}(x,\alpha) = \tilde{F}\big(\tilde{F}(x,\alpha),0\big) = F\big(\tilde{F}(x,\alpha),0\big) = F\big(\tilde{\tilde{F}}(x,\alpha),0\big) =$$
$$= \tilde{\tilde{F}}\big(\tilde{\tilde{F}}(x,\alpha),0\big) = \tilde{\tilde{F}}(x,\alpha).$$

It follows from this equality and from (15) that $\tilde{F} = \tilde{\tilde{F}}$. Since the function of the form (9) is a suitable extension of F, we infer that $\tilde{F}$ must be of the form (9). This completes the proof.

Theorem 3. If $F : E \times \langle 0,\infty) \longrightarrow E$ can be represented in the form (3), where $f_0 : E \longrightarrow E$ is any function, $F^* : f_0(E) \times \langle 0,\infty) \longrightarrow f_0(E)$ satisfies functional equation (1) and for arbitrary $\alpha \in \langle 0,\infty)$, the function $f_0(E) \ni x \longmapsto F^*(x,\alpha)$ is invertible, then the functions f_0 and F^* are determined uniquely.

P r o o f . Let us assume that condition (3) holds. By applying Lemma 1 to the function F^*, we obtain

$$f_0(x) = F^*\big(f_0(x),0\big) = F(x,0).$$

Thus $f_0(x)$ is determined uniquely. It follows from (3) that F^* is also determined uniquely.

T h e o r e m 4. A function $F : E \times (-\infty,\infty) \to E$ satisfies conditions (a), (b), (c) if and only if F is of the form

$$(18) \qquad F(x,\alpha) = \begin{cases} \bar{F}\big(f_0(x),\alpha\big) & \text{for } \alpha \geq 0, \quad x \in E \\ \bar{F}(x,\alpha) & \text{for } \alpha < 0, \ (x,\alpha) \in D_{\bar{F}}, \end{cases}$$

where the function $f_0 : E \to E$ satisfies (4), the function $\bar{F} : f_0(E) \times (-\infty,\infty) \to f_0(E)$ satisfies conditions (a), (b), (c), and for every $\alpha \geq 0$ the function $f_0(E) \ni x \mapsto \bar{F}(x,\alpha)$ is invertible. If $F : E \times (-\infty,\infty) \to E$ can be written in the form (18), then the functions f_0 and $\bar{F}$ are determined uniquely.

P r o o f . Let us assume that $F : E \times (-\infty,\infty) \to E$ satisfies conditions (a), (b), (c). Then the function $F|_{E \times \langle 0,\infty)}$ satisfies condition (A), and hence, by Theorem 1, it is of the form (3). In consequence, as we have shown in the proof ($C \Rightarrow A$) of Theorem 1 $F|_{E \times \langle 0,\infty)}$ can be extended to a function $\tilde{F}$ of the form (9), where the function $\bar{F} : f_0(E) \times (-\infty,\infty) \to f_0(E)$ satisfies conditions (a), (b), (c). Moreover, from (12) it follows that the function $f_0(E) \ni x \mapsto \bar{F}(x,\alpha)$, $\alpha \geq 0$, is invertible. But formula (18) is equivalent to formula (9) (see Remark 1). Thus $\tilde{F}$ is of the form (18) and consequently, in virtue of Theorem 2, F is of the form (18) (as $\tilde{F} = F$).

Now let us assume that the function F is of the form (18). Then, putting: $F^*(x,\alpha) := \bar{F}(x,\alpha)$ for $x \in f_0(E)$, $\alpha \geq 0$ we see that the function $F|_{E \times \langle 0,\infty)}$ can be written in form (3). We have shown in the proof ($C \Rightarrow A$) of Theorem 1 that the function F of the form (9) satisfies condition (a), (b), (c). But formula (9) is equivalent to formula (18) (see Remark 1). Thus the function F satisfies conditions (a), (b), (c).

The proof of the uniqueness of the representation of the function F in the form (18) is similar to that of Theorem 3. This statement completes the proof.

It is immediately seen that in the above considerations the sets $(-\infty,\infty)$ and $\langle 0,\infty)$ may be replaced respectively by an arbitrary subgroup G of the additive group of real numbers and by the set G^+ of non-negative elements of G.

REFERENCES

[1] M. K u c z m a : Functional equations in a single variable. Warszawa 1968.

[2] S. M i d u r a ; J. T a b o r : Sur les iterations avec un paramètre reel des fonctions sous module. Demonstratio Math. 6 (1973) 271-286.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY IN KRAKÓW

Received May 2nd, 1975.