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NON-LINEAR BOUNDARY VALUE PROBLEM OF STATICS IN THE THEORY OF ELASTICITY FOR MULTIPLY CONNECTED DOMAIN

1. Statement of the problem

Let E_3 be the three-dimensional Euclidean space. Suppose that $D_0 \subset E_3$ is a domain bounded by a surface S_0. Further, suppose that inside D_0 there are m disjoint surfaces $S_1, S_2, \ldots, S_m$, each of them being a boundary of a simply connected domain D_i $(i = 1,2,\ldots,m)$, where $D_j \cap D_k = \emptyset$ for $j \neq k$ $(j,k = 1,2,\ldots,m)$. Denote $\bar{D}_k = D_k \cup S_k$ $(k = 0,1,\ldots,m)$, $D^+ = D_0 \setminus \bigcup\limits^{m} \bar{D}_k$, $S = \bigcup\limits_{k=0}^{m} S_k$.

We shall consider the following problem: Find a vector $u(x) = \left[u_1(x), u_2(x), u_3(x)\right]$ satisfying in D^+ the system

$$A\left(\frac{\partial}{\partial x}\right) u(x) = -\Phi\left(x,u(x)\right) \tag{1}$$

and the boundary conditions

$$\left[T\left(\frac{\partial}{\partial z}, n\right) u(z)\right]^+ + \sigma(z)u^+(z) = F^k\left(z,u(z)\right) \tag{2}$$

for $z \in S_k$ $(k = 0,1,\ldots,r;\ 0 \leqslant r < m)$,

$$u^+(z) = f^k(z) \tag{3}$$

for $z \in S_k$ $(k = r+1,\ldots,m)$.

$A\left(\frac{\partial}{\partial x}\right)$, $T\left(\frac{\partial}{\partial z}, n\right)$ are matrix operators of the theory of elasticity: the first is the Kelvin matrix, the second - the stress operator. The elements of these matrices are given by formulae

$$(4) \qquad A_{ij}\left(\frac{\partial}{\partial x}\right) = \mu\delta_{ij}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + (\lambda + \mu)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$(5) \qquad T_{ij}\left(\frac{\partial}{\partial z}, n\right) = \lambda n_i \frac{\partial}{\partial z_j} + \mu n_j \frac{\partial}{\partial z_i} + \mu\delta_{ij}\frac{\partial}{\partial n},$$

where $i,j = 1,2,3$ and λ, μ are Lame constants, δ_{ij} - the Kronecker symbol, $n = [n_1,n_2,n_3]$ - a unit vector normal in a considered point, $\sigma(z)$ is a 3×3 - matrix with elements $\sigma_{ij}(z)$.

$\Phi(x,u)$, $F^k(z,u)$, $f^k(z)$ are vectors with coordinates $\Phi_j(x,u_1,u_2,u_3)$, $F^k_j(z,u_1,u_2,u_3)$, $f^k_j(z)$, $j = 1,2,3$, respectively.

We suppose that

I. The surfaces S_k $(k = 0,1,\dots,m)$ satisfy the Lapunov conditions. One of these conditions concerning the measure of the angle between the normals to S_k in the points y,z has the form

$$(6) \qquad (n_y, n_z) \leqslant C\,|yz|^{\alpha},$$

where $|yz|$ is the Euclidean distance between y and z, C and α are positive constants, $0 < \alpha \leqslant 1$.

II. $\Phi_j(x,u_1,u_2,u_3)$ are real functions defined for $x \in D^+$, $|u_s| \leqslant R$ $(s = 1,2,3)$ satisfying the Hölder-Lipschitz conditions

$$(7) \qquad \left|\Phi_j(x,u_1,u_2,u_3) - \Phi_j(\tilde{x},\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\right| \leqslant K_\Phi |x\tilde{x}|^h + k_\Phi \sum_{s=1}^{3} |u_s - \tilde{u}_s|,$$

where $K_\Phi > 0$, $k_\Phi > 0$, $0 < h < \alpha \leqslant 1$.

III. $f^k_j(z)$ are real functions defined for $z \in S_k$ $(k = r+1,\dots,m)$ satisfying the Hölder conditions

(8) $$\left|f_j^k(z) - f_j^k(\tilde{z})\right| \leqslant K_f\,|z\tilde{z}|^h,$$

where $K_f > 0$.

IV. $F_j^k(z,u_1,u_2,u_3)$ are real functions defined for $z \in S_k$, $|u_s| \leqslant R$ $(s = 1,2,3;\ k = 0,1,\dots,r)$ satisfying the Hölder-Lipschitz conditions

(9) $$\left|F_j^k(z,u_1,u_2,u_3) - F_j^k(\tilde{z},\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\right| \leqslant K_F\left(|z\tilde{z}|^h + \sum_{s=1}^{3}|u_s - \tilde{u}_s|\right),$$

where $K_F > 0$. Moreover, the functions F_j^k are differentiable with respect to u_s $(s = 1,2,3)$ in their domains and their derivatives satisfy the Hölder-Lipschitz conditions

(10) $$\left|\frac{\partial}{\partial u_l}F_j^k(z,u_1,u_2,u_3) - \frac{\partial}{\partial u_l}F_j^k(\tilde{z},\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\right| \leqslant$$
$$\leqslant \tilde{K}_F\left(|z\tilde{z}|^h + \sum_{s=1}^{3}|u_s - \tilde{u}_s|\right). \qquad (l = 1,2,3),$$

where $\tilde{K}_F > 0$.

V. $\sigma_{ij}(z)$ are real functions defined on the surfaces S_k $(k = 0,1,\dots,r)$ satisfying the Hölder conditions

(11) $$\left|\sigma_{ij}(z) - \sigma_{ij}(\tilde{z})\right| \leqslant K_\sigma\,|z\tilde{z}|^h,$$

where $K_\sigma > 0$. Moreover, we suppose that the quadratic form $\sum\sigma_{ij}(z)\,\xi_i\,\xi_j$ is positive definite.

2. The Green tensor

Let S_{m+1} be the closed Lapunov surface which is the boundary of the domain D_{m+1}. Suppose that all surfaces S_k $(k = 0,1,\dots,m)$ lie inside D_{m+1}. Let $D_{m+1}^{(r)} = D_{m+1} \setminus \bigcup_{k=r+1}^{m} \bar{D}_k$. Consider the problem

(12) $$A\left(\frac{\partial}{\partial x}\right)u(x) = 0, \qquad x \in D_{m+1}^{(r)}$$

(13) $$u^+(y) = f^k(y), \quad y \in S_k, \quad k = r+1,\dots,m+1$$

In § 9.4 of the monograph [1] the existence of the Green tensor $G(x,y;\ D_{m+1}^{(r)})$ for the problem (12), (13) has been proved and its properties have been investigated. In § 9.6 the tensor $G\left(x,y;\ D_{m+1}^{(r)}\right)$ has been used to prove the existence and uniqueness of the solution of the problem (1), (2), (3) in the linear case i.e., when the vectors Φ and F^k are independent of the vector u.

In the present paper we shall use the tensor $G(x,y;\ D_{m+1}^{(r)})$ to solve the problem (1), (2), (3). For brevity we shall write $G(x,y)$ instead of $G\left(x,y;\ D_{m+1}^{(r)}\right)$. In the sequel we denote the transpose of a matrix by an asterisk, e.g. $G^*(x,y)$ denotes the transpose of $G(x,y)$. One can prove that $G(x,y) = G^*(y,x)$. We shall use the notation

$$\tilde{G}(x,y) \equiv T\left(\frac{\partial}{\partial y},\ n\right) G^*(x,y). \tag{14}$$

3. The integral equations of the problem

We seek a solution of the problem (1), (2), (3) in the form

$$u(x) = -\frac{1}{2}\sum_{k=r+1}^{m}\int_{S_k} G^*(x,y)f^k(y)dy + \sum_{k=0}^{r}\int_{S_k} G(x,y)\,\varphi(y)dy + \tag{15}$$

$$+\frac{1}{2}\int_{D^+} G(x,y)\Phi(y,u(y))dy, \qquad x \in D^+.$$

Suppose that the coordinates of the vector φ satisfy the Hölder conditions on the surfaces S_k $(k = 0,1,\ldots,r)$. Taking into account the properties of the tensor $G(x,y)$ and substituting (15) into the boundary condition (2), we obtain the following integral equation for the vector φ

$$(16)\quad \varphi(z) + \sum_{k=0}^{r} \int_{S_k} \left[\tilde{G}(z,y) + \sigma(z)G(z,y)\right] \varphi(y)dy =$$

$$= F^k\left[z, -\frac{1}{2} \sum_{l=r+1}^{m} \int_{S_l} \tilde{G}^*(z,y)f^l(y)dy + \sum_{l=0}^{r} \int_{S_l} G(z,y)\varphi(y)dy + \right.$$

$$\left. + \frac{1}{2} \int_{D^+} G(z,y)\phi(y,u(y))dy\right] +$$

$$+ \frac{1}{2}\left[T\left(\frac{\partial}{\partial z}, n\right) + \sigma(z)\right]\left[\sum_{l=r+1}^{m} \int_{S_l} \tilde{G}^*(z,y)f^l(y)dy - \right.$$

$$\left. - \int_{D^+} G(z,y)\,\phi\big(y,u(y)\big)\,dy\right],$$

where $z \in \bigcup_{k=0}^{r} S_k$. Thus we have arrived at the system (15), (16) of strongly singular non-linear integral equations with unknown vectors u and φ. We shall prove the existence and uniqueness of a solution of the system (15), (16) using the Banach fixed-point theorem ([4], p. 37).

4. The functional space

Let X be the set of all systems of real functions

$$U = \left[u_1(x),\ u_2(x),\ u_3(x),\ \varphi_1(z),\ \varphi_2(z),\ \varphi_3(z)\right]$$

defined and continuous for $x \in D^+ \cup S$, $z \in \bigcup_{k=0}^{r} S_k$. Moreover, we suppose that the functions $u_j(x)$, $\varphi_j(z)$ $(j = 1,2,3)$ satisfy the conditions

$$(17)\quad |u_j(x)| \leq R,\ |\varphi_j(z)| \leq \varrho,\ |\varphi_j(z) - \varphi_j(\tilde{z})| \leq K_\varphi\ |z\tilde{z}|^{h_\varphi},$$

where ϱ and K_φ are positive constants which may be chosen arbitrarily and the exponent h_φ is fixed and satisfies the inequality $0 < h_\varphi < h$.

Let $U_i = [u_1^i, u_2^i, u_3^i, \varphi_1^i, \varphi_2^i, \varphi_3^i]$, $i = 1,2$. In the set X we define the distance by the formula

$$(18)\qquad d(U_1,U_2) = \max_j \sup_x \left| u_j^1(x) - u_j^2(x) \right| + \max_j \sup_z \left| \varphi_j^1(z) - \varphi_j^2(z) \right| +$$

$$+ \max_j H\left[\varphi_j^1(z) - \varphi_j^2(z)\right], \quad (j = 1,2,3),$$

where

$$(19)\qquad H\left[\varphi_j(z)\right] = \sup_{z,\tilde z} \frac{|\varphi_j(z) - \varphi_j(\tilde z)|}{|z\tilde z|^{h_\varphi}}.$$

The set X with the distance (18) is a complete metric space. In the space X we introduce the operator $\mathcal{A}$ mapping the points $U = [u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3]$ of this space into the points $V = [v_1, v_2, v_3, \psi_1, \psi_2, \psi_3]$ according to the formulae

$$(20)\qquad v(x) = -\frac{1}{2} \sum_{k=r+1}^{m} \int_{S_k} \tilde G^*(x,y) f^k(y)\,dy + \sum_{k=0}^{r} \int_{S_k} G(x,y)\,\psi(y)\,dy +$$

$$+ \frac{1}{2} \int_{D^+} G(x,y)\,\Phi(y,u(y))\,dy,$$

$$(21)\qquad \psi(z) + \sum_{k=0}^{r} \int_{S_k} \left[\tilde G(z,y) + \sigma(z) G(z,y)\right] \psi(y)\,dy = g(z),$$

where

$$(22)\qquad g(z) = F^k(z,\bar u(z)) + \frac{1}{2}\left[T\left(\frac{\partial}{\partial z}, n\right) + \sigma(z)\right] \left[\sum_{l=r+1}^{m} \int_{S_l} \tilde G^*(z,y) f^l(y)\,dy -\right.$$

$$\left. - \int_{D^+} G(z,y)\,\Phi(y,u(y))\,dy \right],$$

$$(23)\quad \bar{u}(z) = -\frac{1}{2}\sum_{l=r+1}^{m}\int_{S_l}\tilde{G}^*(z,y)f^l(y)dy + \sum_{l=0}^{r}\int_{S_l} G(z,y)\,\varphi(y)dy + $$

$$+\frac{1}{2}\int_{D^+} G(z,y)\,\Phi(y,u(y))\,dy.$$

5. Properties of the operator $\mathcal{A}$

L e m m a 1. If the constant K_F and the least upper bounds of functions $\|\Phi_j(x,u_1,u_2,u_3)\|$ and $\|F_j^k(z,u_1,u_2,u_3)\|$ are sufficiently small and if the constants R,ϱ, K_φ are sufficiently large, then the values of the operator $\mathcal{A}$ belong to the space X.

P r o o f . Let us consider the integral equation (21). Equations of this type have been investigated by V.D.Kupradze (e.g. [1], pp.170-201). From his results it follows that if the vector g(z) satisfies the Hölder condition, then the equation (21) has a unique solution which is given by the formula

$$(24)\qquad \psi(z) = B(z)g(z) + \sum_{k=0}^{r}\int_{S_k} N(z,y)g(y)dy,$$

where B(z) is a matrix whose properties have been investigated in [2], N(z,y) is a matrix resolvent of the equation (21). In order to use formula (24) we must show that the vector g(z) satisfies the Hölder condition. Let $\tilde{z},z \in \bigcup_{k=0}^{r} S_k$. We have

$$(25)\quad g(z) - g(\tilde{z}) = \left[F^k(z,\bar{u}(z)) - F^k(\tilde{z},\bar{u}(\tilde{z}))\right] +$$

$$+\frac{1}{2}\sum_{l=r+1}^{m}\int_{S_l}\left[T\left(\frac{\partial}{\partial z},n\right)\tilde{G}^*(z,y) - T\left(\frac{\partial}{\partial \tilde{z}},n\right)\tilde{G}^*(\tilde{z},y)\right] f^l(y)dy$$

$$+\frac{1}{2}\left[\sigma(z) - \sigma(\tilde{z})\right]\sum_{l=r+1}^{m}\int_{S_l}\tilde{G}^*(z,y)f^l(y)dy +$$

$$+\frac{1}{2}\sigma(\tilde{z})\sum_{l=r+1}^{m}\int_{S_l}\left[\tilde{G}^*(z,y) - \tilde{G}^*(\tilde{z},y)\right]f^l(y)dy +$$

$$+\frac{1}{2}\int_{D^+}\left[T\left(\frac{\partial}{\partial \tilde{z}},n\right)G(\tilde{z},y) - T\left(\frac{\partial}{\partial z},n\right)G(z,y)\right]\Phi(y,u(y))dy +$$

$$+ \frac{1}{2}\left[\sigma(\tilde{z}) - \sigma(z)\right] \int_{D^+} G(\tilde{z},y)\,\Phi(y,u(y))\,dy +$$

$$+ \frac{1}{2}\,\sigma(z) \int_{D^+} \left[G(\tilde{z},y) - G(z,y)\right] \Phi(y,u(y))\,dy.$$

We introduce the following notations

$$M_f = \max_{j,k} \sup_{z \in S_k} \left| f_j^k(z) \right| \qquad (k = r+1,\dots,m;\ j = 1,2,3)$$

$$M_\Phi = \max_{j} \sup_{\substack{x \in D^+ \\ |u_s| \le R}} \left| \Phi_j(x,u_1,u_2,u_3) \right| \qquad (j,s = 1,2,3)$$

$$M_\sigma = \max_{i,j} \sup_{z \in S_k} \left| \sigma_{ij}(z) \right| \qquad (k = 0,1,\dots,r;\ i,j = 1,2,3).$$

Making use of the assumptions and taking into account the well-known properties of surface integrals we obtain from (25) the following inequalities

$$(26)\quad \left| g_j(z) - g_j(\tilde{z}) \right| \le \Big[M_F(a_1+a_2K_\sigma+a_3M_\sigma+a_4K_F) +$$

$$+ M_\Phi(a_5+a_6K_\sigma+a_7M_\sigma+a_8K_F) + K_F(a_9+a_{10}\varrho)\Big]\ |z\tilde{z}|^h, \quad (j=1,2,3),$$

where the constants $a_1,\dots,a_{10}$ depend on Lamé constants. For brevity we write the inequalities (26) in the form

$$(27)\qquad \left| g_j(z) - g_j(\tilde{z}) \right| \le K_g\ |z\tilde{z}|^h, \qquad (j = 1,2,3).$$

Thus the vector $g(z)$ satisfies the Hölder condition.

Analogously as in [2], we introduce the following notations

$$M_B = \max_{i,j} \sup_{z \in S_k} \left| B_{ij}(z) \right| \qquad (k = 0,1,\dots,r;\ i,j = 1,2,3)$$

$$M_N = \max_{i,j} \sup_{z \in S_k} \left| \int_{S_k} N_{ij}(z,y)dy \right| \qquad (k = 0,1,\dots,r;\ i,j = 1,2,3).$$

From formula (24) (see [3] formula (52)) we obtain the estimations

$$(28) \qquad |\psi_j(z)| \leq (M_B + M_N)\, M_g + b_4 K_g , \qquad (j = 1,2,3),$$

where $M_g = M_F + b_1 M_f + b_2 M_f M_6 + b_3 M_\Phi$ and b_1, b_2, b_3, b_4 depend on Lamé constants. We denote $M_\psi = (M_B + M_N)\, M_g + b_4 K_g$.

From results of V.D.Kupradze it follows that the vector $\psi(z)$ satisfies the Hölder condition. Namely, taking into account the inequalities (27) and (28), as well as the estimations (36) and (38) given in [2], we obtain

$$(29) \quad |\psi_j(z) - \psi_j(\tilde z)| \leq (b_5 CM_g + M_B K_g + b_6 K_g)\ |z\tilde z|^{h_\varphi} \qquad (j=1,2,3),$$

where b_5, b_6 depend on Lamé constants.

Next we write the inequalities (29) in the following form

$$(30) \qquad |\psi_j(z) - \psi_j(\tilde z)| \leq K_\psi\ |z\tilde z|^{h_\varphi} , \qquad (j = 1,2,3).$$

From the formula (20) defining the vector v(x) and from the inequality (28) we obtain

$$(31) \qquad |v_j(x)| \leq M_v , \qquad (j = 1,2,3),$$

where $M_v = b_7 M_f + b_8 M_\psi + b_9 M_\Phi$ and b_7, b_8, b_9 depend on Lame constants.

From (28), (30), (31) and the definitions (20), (21) of the operator $\mathcal{A}$ we deduce that the sufficient condition for $\mathcal{A}(U) \in X$ is the system of inequalities

$$(32) \qquad M_v \leq R,\ M_\psi \leq \varrho\ ,\ K_\psi \leq K_\varphi .$$

From assumptions concerning the constants M_Φ, M_F, K_F it follows that there exists $\varrho = \varrho_0$ for which $M_\psi \leq \varrho$. It is easy to see that if ϱ is fixed then the inequalities $M_v \leq R$, $K_\psi \leq K_\varphi$ are satisfied independently of Lamé constants.

L e m m a 2 . If the constants k_Φ, K_F, $\tilde{K}_F$ are sufficiently small and the inequalities (32) are satisfied, then the operator $\mathcal{A}$ is contractive, i.e. there exists a positive constant $\alpha < 1$ such that for every points U_1, U_2 of the space X we have

$$(33) \qquad d(A(U_1),\ A(U_2)) \leq \alpha\, d(U_1, U_2).$$

P r o o f . We denote $A(U_i) = V_i$, $(i = 1,2)$, where

$$V_i = \left[v_1^i, v_2^i, v_3^i, \psi_1^i, \psi_2^i, \psi_3^i\right].$$

To prove inequality (33) we consider the expressions

$$\left|\psi_j^1(z) - \psi_j^2(z)\right|, \quad \left|v_j^1(x) - v_j^2(x)\right|, \quad H\left[\psi_j^1(z) - \psi_j^2(z)\right].$$

The vectors $\psi^1(z)$, $\psi^2(z)$ satisfy the integral equation (21). We substitute them into (21) and subtract, then we get an integral equation which is satisfied by the difference $\psi^1(z) - \psi^2(z)$. The right-hand side of this equation is

$$(34) \qquad \bar{g}(z) = F^k(z, \bar{u}^1(z)) - F^k(z, \bar{u}^2(z)) + - \frac{1}{2}\int_{D^+}\left[T\left(\frac{\partial}{\partial z}, n\right) + \sigma(z)\right] G(z,y)\left[\Phi(y, u^1(y)) + - \Phi(y, u^2(y))\right] dy,$$

where

$$(35) \qquad \bar{u}^i(z) = -\frac{1}{2}\sum_{l=r+1}^{m}\int_{S_l}\tilde{G}^*(z,y) f^l(y)\,dy + \sum_{l=0}^{r}\int_{S_l} G(z,y)\varphi^i(y)\,dy + + \frac{1}{2}\int_{D^+} G(z,y)\Phi(y, u^i(y))\,dy, \qquad (i = 1,2).$$

The vector $\bar{g}(z)$ has a similar form to that defined in [3] (formula (67)). Hence by (77) of [3] we can write the following inequalities

(36) $\left|\bar{g}_j(z) - \bar{g}_j(\tilde{z})\right| \leq$

$$\leq \Big\{k_\Phi\big[c_1+c_2K_6+c_3M_\sigma+c_4K_F+c_5\tilde{K}_F(e_1+e_2M_\Phi+e_3\varrho_0)\ \max_s\ \sup_x\left|u_s^1(x) - u_s^2(x)\right|+$$

$$+\big[c_6K_F+c_7\tilde{K}_F(e_1+e_2M_\Phi+e_3\varrho_0)\big]\max_s\ \sup_z\left|\varphi_s^1(z)-\varphi_s^2(z)\right|\Big\}\ |z\tilde{z}|^{h_\varphi},$$

$$(j,s = 1,2,3),$$

where $c_1,\dots,c_7,e_1,e_2,e_3$ depend on Lamé constants. For brevity we write (36) in the form

(37) $$\left|\bar{g}_j(z) - \bar{g}_j(\tilde{z})\right| \leq \big[k_\Phi M_1\ \max_s\ \sup\ \left|u_s^1 - u_s^2\right| +$$

$$+(c_6K_F+\tilde{K}_FM_2)\ \max_s\ \sup\ \left|\varphi_s^1 - \varphi_s^2\right|\big]\ |z\tilde{z}|^{h_\varphi}, \qquad (j,s=1,2,3).$$

The vector $\bar{g}(z)$ satisfies the Hölder condition, thus the formula (24) with $\psi^1(z) - \psi^2(z)$ instead of $\psi(z)$ and $\bar{g}(z)$ instead of $g(z)$ holds.

Now we may give estimations for $\left|\psi_j^1(z) - \psi_j^2(z)\right|$ similar to (81) in [3]

(38) $$\left|\psi_j^1(z)-\psi_j^2(z)\right| \leq k_\Phi\big[(p_1+p_2K_F)(M_B+M_N)+p_3M_1\big]\ \max_s\ \sup_x\left|u_s^1(x)-u_s^2(x)\right| +$$

$$+\ \big[p_4K_F(M_B+M_N)+p_5(c_6K_F+\tilde{K}_FM_2)\big]\ \max_s\ \sup_z\ \left|\varphi_s^1(z)-\varphi_s^2(z)\right|,\quad (j,s=1,2,3),$$

where $p_1,\dots,p_5$ depend on Lamé constants.

Similarly, using the estimations (83), (89) from [3] we obtain

(39) $\left|v_j^1(x) - v_j^2(x)\right| \leq$

$$\leq k_\Phi\big[p_6+(p_1+p_2K_F)(M_B+M_N)+p_3M_1\big]\ \max_s\ \sup_x\left|u_s^1(x) - u_s^2(x)\right| +$$

$$+p_7\big[p_4K_F(M_B+M_N)+p_5(c_6K_F+\tilde{K}_FM_2)\big]\max_s\ \sup_z\left|\varphi_s^1(z)-\varphi_s^2(z)\right|, \qquad (j,s=1,2,3),$$

$$(40)\quad H\left[\psi_j^1(z)-\psi_j^2(z)\right]\leqslant$$

$$\leqslant K_{\Phi}\left[(p_1+p_2K_F)(p_8C+p_9)+p_{10}M_BM_1\right]\max_s\sup_x\left|u_s^1(x)-u_s^2(x)\right|+$$

$$+\left[K_F(p_{11}C+p_{12})+p_{13}M_B(c_6K_F+\tilde{K}_FM_2)\right]\max_s\sup_z\left|\varphi_s^1(z)-\varphi_s^2(z)\right|,\quad (j,s=1,2,3),$$

where the constants $p_6,\dots,p_{13}$ depend on Lamé constants.

We put

$$(41)\quad \alpha=\max\left\{k_{\Phi}\left[p_6+(p_1+p_2K_F)(2M_B+2M_N+p_8C+p_9)+M_1(2p_3+p_{10}M_B)\right],\right.$$

$$\left.K_F\left[p_4(M_B+M_N)(1+p_7)+p_{11}C+p_{12}+c_6(2p_5+p_{13}M_B)\right]+\tilde{K}_FM_2(2p_5+p_{13}M_b)\right\}.$$

Suppose that the constants k_{Φ}, K_F, $\tilde{K}_F$ are such small that $\alpha<1$. Then from inequalities (38), (39), (40) and from definition (18) we infer that $d(V_1,V_2)\leqslant\alpha\cdot d(U_1,U_2)$. This proves the thesis of the Lemma 2.

6. Solution of the problem

Theorem. If the assumption I-V are satisfied and the constants M_{Φ}, M_F, k_{Φ}, K_F, $\tilde{K}_F$ are sufficiently small, then there exists a unique regular vector of the form (15) satisfying in D^+ the equation (1) and boundary conditions: (2) on the surfaces S_k $(k=0,1,\dots,r)$ and (3) on the surfaces S_k $(k=r+1,\dots,n)$.

Proof. From Banach's fixed-point theorem it follows that if the hypotheses of Lemmas 1 and 2 hold, then the system of integral equations (15), (16) has in the space X a unique solution which we denote by $u^*(x)$, $\varphi^*(x)$. From the representation of the solution in the from (15) and from the properties of Green tensor it follows that $u^*(x)$ satisfies the boundary conditions (2), (3) on suitable surfaces.

Moreover, from the properties of volume integrals (Theorem 2 from [3]) and from the assumptions about the vector $\Phi(x,u(x))$ it follows that there exist second derivatives of the vector $u^*(x)$ and this vector satisfies equation (1) in D^+.

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