DEMONSTRATIO MATHEMATICA
Vol. IX No 1 1976

Zbigniew Olszak

ON TOTALLY UMBILICAL SUBMANIFOLDS OF CARTAN-SYMMETRIC MANIFOLDS

Introduction

In [2] Miyazawa and Chuman have proved the following.

T h e o r e m . Let us suppose that a V_m is a totally umbilical surface immersed in a Cartan - symmetric space. Then, in order that the V_m is also Cartan - symmetric, it is necessary and sufficient that the mean curvature H is a constant.

In this paper, we determine some conditions for a totally umbilical submanifold with mean curvature different from zero, imbedded in a Cartan - symmetric manifold, to be a manifold of constant curvature.

1. Preliminaries

Let V_m be an m - dimensional Riemannian manifold imbedded in an n - dimensional Riemannian manifold V_n, and let $u^\lambda = u^\lambda(w^i)$ be the parametric representation of the submanifold V_m in V_n, where (u^λ) are local coordinates in V_n and (w^i) are local coordinates in V_m. Let $B_i^\lambda = \partial_i u^\lambda$, where $\partial_i = \partial/\partial w^i$.

If $G_{\lambda\omega}$ is the fundamental tensor of the manifold V_n, then g_{ji} defined by $g_{ji} = B_j^\lambda B_i^\omega G_{\lambda\omega}$ is the first fundamental tensor of the submanifold V_m. In the sequel the Greek indices take on values $1,\dots,n$ and the Latin indices take on values $1,\dots,m$ $(m < n)$.

Let N_x^λ $(x = m+1,\dots,n)$ be pairwise orthogonal unit normals to V_m. Then we have the relations

$$(1)\qquad G_{\lambda\omega} N_x^\lambda N_x^\omega = e_x, \quad G_{\lambda\omega} N_x^\lambda N_y^\omega = 0 \ (x \neq y), \quad G_{\lambda\omega} N_x^\lambda B_i^\omega = 0,$$

where e_x is the indicator of the vector N_x^λ.

The Schouten's curvature tensor H_{ji}^λ of the submanifold V_m is defined by

$$(2)\qquad H_{ji}^{\lambda} = \nabla_j B_i^{\lambda},$$

where ∇_j denotes covariant differentiation with respect to the fundamental tensor g_{ji} of V_m.

If we put

$$(3)\qquad H_{ji}^{\lambda} = \sum_x e_x H_{jix} N_x^{\lambda},$$

then the second fundamental tensor H_{jix} for N_x is given by

$$(4)\qquad H_{jix} = H_{ji}^{\lambda} N_{x\lambda},$$

where $N_{x\lambda} = N_x^{\omega} G_{\omega\lambda}$.

The Gauss and Codazzi equations for V_m can be written in the form

$$(5)\qquad R_{lkji} = \bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} B_i^{\omega} + \sum_x e_x (H_{lix} H_{kjx} - H_{ljx} H_{kix})$$

and

$$(6)\qquad \bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} N_x^{\omega} = \nabla_l H_{kjx} - \nabla_k H_{ljx} + \sum_y e_y (L_{lyx} H_{kjy} - L_{kyx} H_{ljy})$$

respectively ([1]), where L_{ixy} defined by

$$(7)\qquad L_{ixy} = (\nabla_i N_x^{\lambda}) N_{y\lambda} \ (= - L_{iyx})$$

is the third fundamental tensor with respect to the normals N_x^λ, and R_{lkji}, $\bar{R}_{\lambda\mu\nu\omega}$ are curvature tensors for V_m and V_n, respectively.

We have also the equations of Weingarten

(8) $$\nabla_i N_x{}^\lambda = - H_i{}^r{}_x B_r^\lambda + \sum_y e_y L_{ixy} N_y{}^\lambda ,$$

where $H_i{}^j{}_x = H_{irx} g^{rj}$.

Let C^λ be a vector field defined along V_m and orthogonal to V_m, then

(9) $$C^\lambda = \sum_x e_x C_x N_x{}^\lambda ,$$

and consequently by (8)

(10) $$\nabla_i C^\lambda = \sum_x e_x (\partial_i C_x) N_x{}^\lambda + \sum_x e_x C_x \left(-H_i{}^r{}_x B_r^\lambda + \sum_y e_y L_{ixy} N_y{}^\lambda\right) =$$

$$= - \sum_x e_x H_i{}^r{}_x C_x B_r^\lambda + \sum_x e_x (\partial_i C_x + \sum_y e_y L_{iyx} C_y) N_x{}^\lambda .$$

Thus, if we put

(11) $$'\nabla_i C_x = (\nabla_i C^\lambda) N_{x\lambda},$$

we get from (10)

(12) $$'\nabla_i C_x = \partial_i C_x + \sum_y e_y L_{iyx} C_y .$$

If $\nabla_i C^\lambda$ is tangent to V_m, that is, if $'\nabla_i C_x = 0$, we say that C^λ is parallel.

L e m m a 1. Let us suppose that C^λ is a vector field defined along V_m and orthogonal to V_m. If C^λ is parallel, then $C_\lambda C^\lambda$ = constant.

P r o o f . From (9) and (1) we find

$$C_\lambda C^\lambda = \sum_x e_x (C_x)^2 .$$

Since ${}'\nabla_i C_x = 0$, from (12) we get

$$\partial_i C_x = -\sum_y e_y L_{iyx} C_y.$$

Thus, by the above equations and (7), we have

$$\partial_i (C_\lambda C^\lambda) = 2\sum_x e_x C_x \partial_i C_x = -2\sum_x\sum_y e_x e_y C_x C_y L_{iyx} = 0,$$

as desired.

If V_m is a submanifold of codimension 1, then the third fundamental tensor vanishes, and consequently the following lemma holds good.

L e m m a 2. Let us suppose that C^λ is a vector field defined along V_m and orthogonal to V_m, and that codim $V_m = 1$. Then, $C_\lambda C^\lambda$ = constant if and only if C^λ is parallel.

2. <u>A totally umbilical submanifold</u>

If $H_{ji}{}^\lambda$ defined by (2) satisfies the relation

$$H_{ji}{}^\lambda = g_{ji} H^\lambda, \tag{13}$$

where the vector H^λ, called the mean curvature vector, is given by

$$H^\lambda = \frac{1}{m} g^{ji} H_{ji}{}^\lambda, \tag{14}$$

then V_m is called a totally umbilical submanifold.

We assume that the V_m is a totally umbilical submanifold.

Putting $H_x = H^\lambda N_{x\lambda}$, we obtain from (4) by (13)

$$H_{jix} = g_{ji} H_x, \tag{15}$$

and from (3) by (14)

$$H^\lambda = \sum_x e_x H_x N_x{}^\lambda . \tag{16}$$

Hence, using (1), we have

$$H_\lambda H^\lambda = \sum_x e_x \,(H_x)^2. \tag{17}$$

The mean curvature H of V_m, i.e. the scalar H such that $H^2 = \left|\sum_x e_x \,(H_x)^2\right|$, may be then written as

$$H^2 = \left|H_\lambda H^\lambda\right|. \tag{18}$$

Substituting (15) and (17) into (5) we see that

$$R_{lkji} = \bar{R}_{\lambda\mu\nu\omega} B_l^{\ \lambda} \; B_k^{\ \mu} \; B_j^{\ \nu} \; B_i^{\ \omega} + H_\lambda H^\lambda \,(g_{li}\, g_{kj} - g_{lj}\, g_{ki}), \tag{19}$$

and substituting (15) into (6) we see that

$$\bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} \; B_j^{\ \mu} \; B_i^{\ \nu} \; N_x^{\ \omega} = g_{ji}\, \nabla_k H_x - g_{ki}\, \nabla_j H_x + \sum_y e_y \, H_y \,(L_{kyx}\, g_{ji} - L_{jyx}\, g_{ki}). \tag{20}$$

From this, using (16), (17) and (7), we obtain

$$\bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} \; B_j^{\ \mu} \; B_i^{\ \nu} \; H^\omega = \frac{1}{2}\, \nabla_k (H_\lambda H^\lambda) g_{ji} - \frac{1}{2}\, \nabla_j (H_\lambda H^\lambda) g_{ki}. \tag{21}$$

The assumption that the mean curvature H of V_m is a constant implies that (21) may be written in the form

$$\bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} \; B_j^{\ \mu} \; B_i^{\ \nu} \; H^\omega = 0. \tag{22}$$

3. A totally umbilical submanifold of a Cartan-symmetric manifold

A Riemannian manifold is called Cartan-symmetric, if its curvature tensor $\bar{R}_{\lambda\mu\nu\omega}$ satisfies

$$\nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega} = 0, \tag{23}$$

where ∇_ε denotes the covariant differentiation with respect to the metric.

In the sequel we suppose that the manifold V_n is Cartan-symmetric, and that the submanifold V_m of V_n is a totally umbilical submanifold, and $n > m > 2$.

We first prove

L e m m a 3. Let V_m be a totally umbilical submanifold with the mean curvature $H \neq 0$, of a Cartan-symmetric manifold V_n. Then, V_m is of constant curvature if and only if $H = const.$ and

(24) $$A_{ji} = A\, g_{ji},$$

where

(25) $$A_{ji} = \sum_x e_x\, ('\nabla_j X_x)('\nabla_i H_x), \qquad A = \frac{1}{m} g^{ji} A_{ji},$$

and $'\nabla_j$ is defined in (11).

P r o o f . Suppose that H = constant. In this case (22) holds. We differentiate (22) covariantly with respect to w^i and by (23), (13) we find

(26) $$g_{lk} \bar{R}_{\lambda\mu\nu\omega} H^\lambda B_j^{\ \mu} B_i^{\ \nu} H^\omega + g_{lj} \bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} H^\mu B_i^{\ \nu} H^\omega + \\ + g_{li} \bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} B_j^{\ \mu} H^\nu H^\omega + \bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} B_j^{\ \mu} B_i^{\ \nu} \nabla_l H^\omega = 0.$$

But by putting $C^\lambda = H^\lambda$, $C_x = H_x$ (12), (15) and (17) into (10), we have

(27) $$\nabla_l H^\omega = - H_\lambda H^\lambda B_l^{\ \omega} + \sum_x e_x\, ('\nabla_l H_x)\, N_x^{\ \omega}.$$

From (26) with the aid of (27) we obtain

$$g_{lk} \bar{R}_{\lambda\mu\nu\omega} H^\lambda B_j^{\ \mu} B_i^{\ \nu} H^\omega - g_{lj} \bar{R}_{\mu\lambda\nu\omega} H^\mu B_k^{\ \lambda} B_i^{\ \nu} H^\omega + \\ - H_\rho H^\rho \bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} B_j^{\ \mu} B_i^{\ \nu} B_l^{\ \omega} + \\ + \sum_x e_x\, ('\nabla_l H_x)\, \bar{R}_{\lambda\mu\nu\omega} B_k^{\ \lambda} B_j^{\ \mu} B_i^{\ \nu} N_x^{\ \omega} = 0,$$

and by (19), (20) and (12)

$$(28)\quad -g_{lk}\, \bar{R}_{\lambda\mu\nu\omega}\, H^{\lambda} B_j^{\mu} B_i^{\nu} H^{\omega} - g_{lj}\, \bar{R}_{\mu\lambda\nu\omega}\, H^{\mu} B_k^{\lambda} B_i^{\nu} H^{\omega} +$$

$$- H_{\varrho} H^{\varrho}\left[R_{kjil} - H_{\lambda} H^{\lambda} (g_{kl}\, g_{ji} - g_{ki}\, g_{jl})\right] +$$

$$+ \sum e_x\, ('\nabla_l H_x) \left[('\nabla_k H_x)\, g_{ji} - ('\nabla_j H_x)\, g_{ki}\right] = 0.$$

Contracting the equation (28) with g^{lk} and using the denotations (25), we get

$$(29)\qquad \bar{R}_{\lambda\mu\nu\omega}\, H^{\lambda} B_j^{\mu} B_i^{\nu} H^{\omega} = H_{\varrho} H^{\varrho}\left(\frac{1}{m-1} R_{ji} - H_{\lambda} H^{\lambda} g_{ji}\right) +$$

$$+ \frac{1}{m-1} (A_{ji} - m\, A\, g_{ji}).$$

We substitute (29) into (28) and find

$$(30)\quad - H_{\varrho} H^{\varrho} \left[R_{kjil} - \frac{1}{m-1} (g_{kl}\, R_{ji} - g_{jl}\, R_{ki})\right] =$$

$$= \frac{m}{m-1} A\, (g_{lk}\, g_{ji} - g_{lj}\, g_{ki}) - \frac{1}{m-1} (g_{kl}\, A_{ji} - g_{jl}\, A_{ki}) -$$

$$- A_{kl}\, g_{ji} + A_{jl}\, g_{ki}.$$

Let now $A_{ji} = A\, g_{ji}$ holds, as desired in (24), and $H = \text{const.} \neq 0$. Then, since by (18) $H_{\varrho} H^{\varrho} = \text{const.} \neq 0$, from (30) it follows

$$(31)\qquad R_{kjil} = \frac{1}{m-1} (g_{kl}\, R_{ji} - g_{jl}\, R_{ki}).$$

We can easily verify that then V_m is a manifold of constant curvature, i.e. that the condition

$$R_{kjil} = \frac{R}{m(m-1)} (g_{kl}\, g_{ji} - g_{ki}\, g_{jl})$$

is satisfied.

Conversely, if V_m is of constant curvature, then (31) is satisfied. Moreover, V_m is then Cartan-symmetric, and by the theorem presented in the introduction the mean curvature H = = constant. Then (30) yields

$$\frac{m}{m-1} A\,(g_{lk}\,g_{ji} - g_{lj}\,g_{ki}) - \frac{1}{m-1}\,(g_{kl}\,A_{ji} - g_{jl}\,A_{ki}) +$$

$$- A_{kl}\,g_{ji} + A_{jl}\,g_{ki} = 0.$$

Summing this equation with g^{ji}, we find $A_{lk} = A\,g_{lk}$, which completes the proof.

In the case codim $V_m = 1$, the assumption H = constant implies by Lemma 2 that the condition (24) is satisfied. Thus we have

T h e o r e m 1. Let V_m $(m = n-1)$ be a totally umbilical submanifold with the mean curvature $H \neq 0$, of a Cartan-symmetric manifold V_n. Then, V_m is of constant curvature if and only if H = constant.

As a consequence of Lemma 3, we obtain

T h e o r e m 2. Let V_m be a totally umbilical submanifold with the mean curvature $H \neq 0$, of a Cartan-symmetric manifold V_n. Then, if $\sum_x e_x('\nabla_j H_x)('\nabla_i H_x) = 0$ and H = const., V_m is of constant curvature.

Furthermore, $'\nabla_j H_x = 0$ also implies the condition (24) and by Lemma 1 it implies H = constant. Hence, we may formulate the following

T h e o r e m 3. Let V_m be a totally umbilical submanifold with the mean curvature $H \neq 0$, of a Cartan-symmetric manifold V_n. Then, if the mean curvature vector H^λ is parallel, V_m is of constant curvature.

REFERENCES

[1] L.P. E i s e n h a r t : Riemannian geometry. Princeton 1949.
[2] T. M i y a z a w a , G. C h ū m a n : On certain subspaces of Riemannian recurrent spaces, Tensor 23 (1972) 253-260.

INSTITUTE OF MATHEMATICS AND TEORETICAL PHYSICS, TECHNICAL UNIVERSITY OF WROCŁAW

Received April 7, 1975.