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ON THE CONVERGENCE OF AN ITERATIVE SEQUENCE TO THE SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENTS

In the theory of optimal control, very frequently there appears the problem of solving a system of ordinary differential equations of second order with deviated arguments. This problem is of great importance for many branches of this theory. Therefore it has been dealt with by many authors. A large number of references can be found e.g. in [3] and [8]. This paper is a generalization of [4]. We consider a system of ordinary differential equations of second order with deviated arguments (1), similarly to the way it was done in [5], but with one difference. Namely, our boundary value conditions are different from those of [5] where authors supposed that initial functions were given. Here, together with equation (1) we shall consider a linear two point boundary value condition. A similar condition was considered in [2] p. 50. In this note initial functions, which authors of paper [5] supposed to be given, are constructed in order that the solution of system (1) fulfil condition (4). We shall prove existence and uniqueness of the solution of problem (1), (4) in certain interval provided that this interval is small. This task is more general than that from paper [1], where the arguments were not deviated. This, however, does not yield better results.

Consider the system of ordinary differential equations of second order with deviated arguments

$$x''(t) = f\Big(t, x(w_1(t)),\dots,x(w_k(t)), x'(w_1(t)),\dots,x'(w_k(t)), \quad (1)$$
$$x''(w_1(t)),\dots,x''(w_k(t))\Big) ,$$

where $x \in R^m$, $t \in D = [a_1, a_2]$, $-\infty < a_1 < a_2 < \infty$, and $f = (f_1,\dots,f_m)$ is mapping of the type

$$f : D \times R^{3mk} \to R^m. \quad (2)$$

The real functions w_i $(i = 1,2,\dots,k)$ of the form

$$w_i : D \to D \quad (3)$$

are continuous in interval D.

Let j denote the identity function in D, i.e. $j(t) = t$. The function f appearing in equation (1) may be treated as an operation mapping the set of functions differentiable twice and defined in interval D into the set of continuous functions defined in the same interval.

Considering (1) as an equation of two functions, we have

$$x'' = f\big(j, x(w_1),\dots,x(w_k), x'(w_1),\dots,x'(w_k), x''(w_1),\dots,x''(w_k)\big).$$

The function on the right-hand side of the above equation is the image of the function x under the operation f. We shall denote it simply by $f[x]$, while the value of this function at a point t will be denoted by $f[x](t)$.

In the whole paper, the symbols $|\cdot|$ and $\|\cdot\|$ will denote the absolute value and a certain norm in the space R^m, respectively. For a continuous function x, defined in D, value of which lie in R^m, we introduce the norm

$$|||x||| := \max_{t \in D} \|x(t)\|.$$

Let $t_i \in D$ $(i = 1,2)$ and $t_1 \le t_2$. We shall seek solutions of the system (1) in interval D in the class of continuous functions with derivatives up to second order. Each of these solutions should satisfy the conditions

(4) $$\begin{cases} \alpha_0 x(t_1) + \alpha_1 x'(t_1) + \alpha_2 x''(t_1) = r_1 \\ \beta_0 x(t_2) + \beta_1 x'(t_2) + \beta_2 x''(t_2) = r_2, \end{cases}$$

where α_j, $\beta_j \in R$ $(j = 0,1,2)$, $x(t_i)$, $x'(t_i)$, $x''(t_i)$, $r_i \in R^m$ for $i = 1,2$. Moreover, assume that

(5) $$\Delta = \begin{vmatrix} \alpha_0, & \alpha_1 \\ \beta_0, & \beta_0(t_2 - t_1) + \beta_1 \end{vmatrix} \neq 0.$$

Relation (5) is true e.g. for the Cauchy condition and for the two-point boundary condition of the first kind.

Now, we shall prove some lemmas.

L e m m a 1. Let the functions w_i $(i = 1,2,\dots,k)$ of the type (3) be continuous and let the function f be of type (2) and uniformly Lipschitz

(6) $$\|f(t,\tilde{z}_1,\dots,\tilde{z}_k,\tilde{\bar{z}}_1,\dots,\tilde{\bar{z}}_k,\tilde{\bar{\bar{z}}}_1,\dots,\tilde{\bar{\bar{z}}}_k) + \\ - f(t,z_1,\dots,z_k,\bar{z}_1,\dots,\bar{z}_k,\bar{\bar{z}}_1,\dots,\bar{\bar{z}}_k)\| \leq \\ \leq \sum_{i=1}^{k} p_i \|\tilde{z}_i - z_i\| + \sum_{i=1}^{k} \bar{p}_i \|\tilde{\bar{z}}_i - \bar{z}_i\| + \sum_{i=1}^{k} \bar{\bar{p}}_i \|\tilde{\bar{\bar{z}}}_i - \bar{\bar{z}}_i\|$$

with non-negativ constants p_i, $\bar{p}_i$, $\bar{\bar{p}}_i$ $(i = 1,2,\dots,k)$. Then the following inequality holds

(7) $$|||f[x_1] - f[x_2]||| \leq Q[x_1,x_2],$$

where the function Q is defined as follows

(8) $$Q[x_1,x_2] = p\,|||x_1 - x_2||| + \bar{p}\,|||x_1' - x_2'||| + \bar{\bar{p}}\,|||x_1'' - x_2''|||,$$

where

where

$$p = \sum_{i=1}^{k} p_i, \quad \bar{p} = \sum_{i=1}^{k} \bar{p}_i, \quad \bar{\bar{p}} = \sum_{i=1}^{k} \bar{\bar{p}}_i .$$

P r o o f . Since the function f is uniformly Lipschitz, we have

$$|||f[x_1] - f[x_2]||| \leqslant \max_{u \in D} \Big[\sum_{i=1}^{k} p_i \| x_1(w_i(u)) - x_2(w_i(u)) \| +$$

$$+ \sum_{i=1}^{k} \bar{p}_i \| x_1'(w_i(u)) - x_2'(w_i(u)) \| + \sum_{i=1}^{k} \bar{\bar{p}}_i \| x_1''(w_i(u)) +$$

$$- x_2''(w_i(u)) \| \Big] .$$

Further, making use of the definition of the norm, we obtain

$$\max_{u \in D} \Big[\sum_{i=1}^{k} p_i \| x_1(w_i(u)) - x_2(w_i(u)) \| + \sum_{i=1}^{k} \bar{p}_i \| x_1'(w_i(u)) +$$

$$- x_2'(w_i'(u)) \| + \sum_{i=1}^{k} \bar{\bar{p}}_i \| x_1''(w_i(u)) - x_2''(w_i(u)) \| \Big] \leqslant \sum_{i=1}^{k} p_i |||x_1(w_i) +$$

$$- x_2(w_i) ||| + \sum_{i=1}^{k} \bar{p}_i ||| x_1'(w_i) - x_2'(w_i) ||| + \sum_{i=1}^{k} \bar{\bar{p}}_i ||| x_1''(w_i) +$$

$$- x_2''(w_i) ||| .$$

Using relation (3) for functions w_i, $i = 1,2,\ldots,k$ and definition (7), we get

$$\sum_{i=1}^{k} p_i ||| x_1(w_i) - x_2(w_i) ||| + \sum_{i=1}^{k} \bar{p}_i ||| x_1'(w_i) - x_2'(w_i) ||| +$$

$$+ \sum_{i=1}^{k} \bar{\bar{p}}_i ||| x_1''(w_i) - x_2''(w_i) ||| \leqslant Q[x_1, x_2],$$

which completes the proof of the lemma.

L e m m a 2. Under the assumption of Lemma 1, the following inequality is valid

$$\Big\| \underbrace{\int_{t_1}^{t}\int_{t_1}^{s_{p-1}} \cdots \int_{t_1}^{s_1}}_{p\ \text{times}} \Big\{ f[x_1](u) - f[x_2](u)\Big\} du\, ds_1 \ldots ds_{p-1} \Big\| \leq d^p\, Q[x_1,x_2],$$

$p = 1,2,\ldots,$ where d denotes the lenght of interval D.

P r o o f . Using p times the elementary inequality for the norm of an integral we obtain the assertion from Lemma 1.

L e m m a 3. Let

$$(9)\quad z_i(t) = \int_{t_1}^{t}\Big\{\int_{t_1}^{s} f[x_i](u)du\Big\}ds + A[x_i](t-t_1) + B[x_i],\quad i=1,2,$$

be functions defined for $t \in D$, where the continuous functions w_i, $i = 1,2,\ldots,k$, fulfil relation (3), the function f is continuous with respect to the system of variables $(t, z_1,\ldots, z_k, \bar z_1,\ldots,\bar z_k, \bar{\bar z}_1,\ldots,\bar{\bar z}_k)$, the constant vectors $A[x_i]$, $B[x_i]$ belong to R^m and the functions x_1, x_2 are continuous together with their derivatives up to second order in the interval D. In order that functions z_i, $i = 1,2$, fulfil boundary value condition (4) with assumption (5), it is necessary that vectors $A[x_i]$, $B[x_i]$, $i = 1,2$, be defined by formulae

$$(10)\quad A[x_i] = \frac{1}{\Delta}\Big\{\alpha_0\big(r_2 - \beta_2 f[x_i](t_2)\big) - \beta_0\big(r_1 - \alpha_2 f[x_i](t_1)\big) +$$

$$- \alpha_0\beta_0 \int_{t_1}^{t_2}\Big(\int_{t_1}^{s} f[x_i](u)du\Big)ds - \alpha_0\alpha_1\int_{t_1}^{t_2} f[x_i](u)du\Big\},$$

$$(11)\quad B[x_i] = \frac{1}{\Delta}\Big\{\beta_0(t_2 - t_1)\,\big(r_1 - \alpha_2 f[x_i](t_1)\big) +$$

$$+ \beta_1\big(r_1 - \alpha_2 f[x_i](t_1)\big) - \alpha_1\big(r_2 - \beta_2 f[x_i](t_2)\big) +$$

$$+ \alpha_1\beta_1\int_{t_1}^{t_2} f[x_i](u)du + \alpha_1\beta_0\int_{t_1}^{t_2}\Big(\int_{t_1}^{s} f[x_i](u)du\Big)ds\Big\},$$

for i = 1,2.

P r o o f . Differentiating both sides of the equation (9) with respect to variable t, we obtain

$$z_i'(t) = \int_{t_1}^{t} f[x_i](u)du + A[x_i], \quad i = 1,2, \tag{12}$$

and

$$z_i''(t) = f[x_i](t), \quad i = 1,2. \tag{13}$$

Inserting (9), (12) and (13) to (4) we get a system of equations with unknown vectors $A[x_i]$ and $B[x_i]$, the solution of which is expressed by formulae (10) and (11). Assumption (5) is necessary for the existence of a unique solution to the mentioned system. This completes the proof of the lemma.

L e m m a 4. Under the assumptions of Lemma 3, the following inequality holds

$$\| A[x_1] - A[x_2] \| \leq C_1 \, Q[x_1,x_2], \tag{14}$$

where $A[x_i]$ is defined by formula (10) for uniformly Lipschitz function f and the constant C_1 is defined by the formula

$$C_1 = \frac{1}{|\Delta|}\left\{ |\alpha_0\beta_2| + |\alpha_2\beta_0| + |\alpha_0\beta_0| d^2 + |\alpha_0\alpha_1| d\right\}. \tag{15}$$

P r o o f . Using (10) and properties of the norm we obtain the inequality

$$\| A[x_1] - A[x_2]\| \leq \frac{1}{|\Delta|}\Big\{ |\alpha_0\beta_2| \; \| f[x_1](t_2) - f[x_2](t_2)\| +$$

$$+|\beta_0\alpha_2| \; \| f[x_1](t_1) - f[x_2](t_1)\| + |\alpha_0\beta_0| \; \Big\| \int_{t_1}^{t_2}\Big\{ \int_{t_1}^{s} \big(f[x_1](u) +$$

$$- f[x_2](u)\big) du \Big\} ds \Big\| + |\alpha_0\alpha_1| \; \Big\| \int_{t_1}^{t_2} \big(f[x_1](u) - f[x_2](u)\big) du \Big\| .$$

Now, the properties of the function "max" and Lemma 2 for cases p = 1 and p = 2 yield the assertion.

L e m m a 5. Let the assumptions of Lemma 3 be fulfilled and let the function f satisfy the Lipschitz condition (6). Then the following inequality holds

$$\| B[x_1] - B[x_2] \| \leq C_2\, Q[x_1,x_2], \tag{16}$$

where the constant C_2 is defined by the formula

$$C_2 = \frac{1}{|\Delta|}\Big\{ |\beta_0\alpha_2|\, d + |\beta_1\alpha_2| + |\alpha_1\beta_2| + |\alpha_1\beta_0|\, d^2 + |\alpha_1\beta_1|\, d \Big\}. \tag{17}$$

In the proof one uses definition (12) and proceeds similarly to the proof of Lemma 4.

L e m m a 6. Let the function f fulfil the Lipschitz condition (6) and let the assumptions of Lemma 3 be fulfilled. Then the following inequality holds

$$|\!|\!| z_1 - z_2 |\!|\!| \leq C_3\, Q[x_1,x_2], \tag{18}$$

where

$$C_3 = d^2 + C_1 d + C_2. \tag{19}$$

P r o o f . Using (9) and the properties of norm we have

$$\| z_1(t) - z_2(t) \| \leq \int_{t_1}^{t} \Big\{ \int_{t_1}^{s} \| f[x_1](u) - f[x_2](u) \|\, du \Big\}\, ds +$$

$$+ \| A[x_1] - A[x_2] \|\, |t - t_1| + \| B[x_1] - B[x_2] \|, \quad t \in D.$$

Now, in view of Lemmas 4 and 5 together with Lemma 2 in case $p = 2$, we obtain the assertion.

L e m m a 7 . Under the assumptions of Lemma 6 the following inequality holds

$$|\!|\!| z_1' - z_2' |\!|\!| \leq C_4\, Q[x_1,x_2], \tag{20}$$

where

$$C_4 = d + C_1. \tag{21}$$

P r o o f . Using (12) and the properties of norm we obtain

$$\|z_1'(t) - z_2'(t)\| \leq \int_{t_1}^{t} \|f[x_1](u) - f[x_2](u)\| \, du +$$

$$+ \| A[x_1] - A[x_2] \| ,$$

$t \in D$, and now Lemma 2 in case $p = 1$ gives the assertion.

The last of this series of lemmas is the following.

L e m m a 8. Under the assumption of Lemma 6 the following inequality is valid

$$\||| z_1'' - z_2'' ||| \leq Q[x_1, x_2]. \tag{22}$$

In the proof one makes use of definitions (13) and (8) and the properties of norm.

Now, we pass to the main theorem of this paper. The problem (1), (4) will be solved with the help of the method of successive approximations. The constructive proof of this theorem makes it possible to apply this method to solving the problem numerically with the use of computers.

T h e o r e m 1. Let the continuous functions w_i, $i = 1,2,\dots,k$, $t \in D$, be of form (3) and let the function f of the type (2) fulfil Lipschitz condition (6) with non-negativ constants p_i, $\bar{p}_i$, $\bar{\bar{p}}_i$, $i = 1,2,\dots,k$, which satisfy the inequality

$$q < 1, \tag{23}$$

where

$$q = C_3\, p + C_4\, \bar{p} + \bar{\bar{p}} .$$

Then under the assumption (5), in the interval D there exists a unique solution of the equation (1) determined by the boundary value condition (4). This solution and its derivative is a uniform limit of the functional sequence $\{x_n\}$ defined by formulae (24) - (28). The rate of this convergence is determined as follows

$$(24)\qquad \begin{cases} |||x_n - x||| \leqslant C_5\, q^n \\ |||x_n' - x'||| \leqslant C_6\, q^n \\ |||x_n'' - x''||| \leqslant C_7\, q^n, \end{cases}$$

where C_i, $i = 5,6,7$, are constant.

P r o o f . To prove our assertion, we shall define three functional sequences (25), (28) and (29) together with two vector sequences (26) and (27). We shall prove that the functional sequences are uniformly convergent and that vector sequences are convergent with respect to each coordinate towards suitable coordinates of limit vectors A and B. Let $x_0 : D \longrightarrow R^m$ be an arbitrary function of the class C^2. Let us form the sequence of functions $\{x_n\}$ defined in interval D by the formula

$$(25)\quad x_n(t) = \int_{t_1}^{t}\int_{t_1}^{s} f[x_{n-1}](u)du\,ds + A[x_{n-1}](t - t_1) + B[x_{n-1}]$$

for $n = 1,2,\dots$ In order that all the functions (25) fulfil condition (4) by Lemma 3 it is necessary to define vectors $A[x_{n-1}]$, $B[x_{n-1}]$ by the following formulae

$$A[x_{n-1}] = \frac{1}{\Delta}\Big\{\alpha_0\big(r_2 - \beta_2 f[x_{n-1}](t_2)\big) +$$

$$(26)\quad - \beta_0\big(r_1 - \alpha_2 f[x_{n-1}](t_1)\big) + \alpha_0\beta_0\int_{t_1}^{t_2}\Big(\int_{t_1}^{s} f[x_{n-1}](u)du\Big)ds +$$

$$- \alpha_0\alpha_1\int_{t_1}^{t_2} f[x_{n-1}](u)du\Big\},$$

$$B[x_{n-1}] = \frac{1}{\Delta}\Big\{\beta_0(t_2 - t_1)\big(r_1 - \alpha_2 f[x_{n-1}](t_1)\big) +$$

$$+ \beta_1(r_1 - \alpha_2 f[x_{n-1}](t_1)) - \alpha_1(r_2 - \beta_2 f[x_{n-1}](t_2)) +$$

$$+ \alpha_1\beta_1 \int_{t_1}^{t_2} f[x_{n-1}](u)du + \alpha_1\beta_0 \int_{t_1}^{t_2}\Big(\int_{t_1}^{s} f[x_{n-1}](u)du\Big)ds\Big\},$$

for $n = 1,2,\ldots$ Then

(28) $$x_n(t) = \int_{t_1}^{t} f[x_{n-1}](u)du + A[x_{n-1}], \quad t \in D$$

and

(29) $$x_n''(t) = f[x_{n-1}](t), \quad t \in D,$$

for $n = 1,2,\ldots$
The functions (25) fulfil the assumptions of Lemma 6. Therefore we obtain

(30) $$|||x_n - x_{n-1}||| \leq C_3\, Q[x_{n-1}, x_{n-2}], \quad n = 1,2,\ldots,$$

where the constant C_3 is defined by the formula (19) and the function Q by formula (8). Similarly the functions (28) satisfy assumptions of Lemma 7, whence

(31) $$|||x_n' - x_{n-1}'||| \leq C_4\, Q[x_{n-1}, x_{n-2}], \quad n = 1,2,\ldots,$$

where C_4 is defined by (21). Formula (29) together with Lemma 8 gives

(32) $$|||x_n'' - x_{n-1}''||| \leq Q[x_{n-1}, x_{n-2}], \quad n = 1,2,\ldots$$

Putting

(33) $$\begin{cases} \tilde{x}_n = |||x_n - x_{n-1}||| \\ \tilde{x}_n' = |||x_n' - x_{n-1}'||| \\ \tilde{x}_n'' = |||x_n'' - x_{n-1}''||| \quad \text{for} \quad n = 1,2,\ldots \end{cases}$$

and making use of definition (8) we have

$$(34)\qquad \begin{cases} \tilde{x}_n \leqslant C_3\, p\, \tilde{x}_{n-1} + C_3\, \bar{p}\, \tilde{x}'_{n-1} + C_3\, \bar{\bar{p}}\, \tilde{x}''_{n-1}, \\ \tilde{x}'_n \leqslant C_4\, p\, \tilde{x}_{n-1} + C_4\, \bar{p}\, \tilde{x}'_{n-1} + C_4\, \bar{\bar{p}}\, \tilde{x}''_{n-1}, \\ \tilde{x}''_n \leqslant p\, \tilde{x}_{n-1} + \bar{p}\, \tilde{x}'_{n-1} + \bar{\bar{p}}\, \tilde{x}''_{n-1} \end{cases}$$

for $n = 1,2,\ldots$

Let us form the new number-valued sequences $\{x_n\}$, $\{x'_n\}$ and $\{x''_n\}$ by the formulae

$$(35)\qquad \begin{cases} \tilde{\tilde{x}}_n = C_3\, p\, \tilde{\tilde{x}}_{n-1} + C_3\, \bar{p}\, \tilde{\tilde{x}}'_{n-1} + C_3\, \bar{\bar{p}}\, \tilde{\tilde{x}}''_{n-1}, \\ \tilde{\tilde{x}}'_n = C_4\, p\, \tilde{\tilde{x}}_{n-1} + C_4\, \bar{p}\, \tilde{\tilde{x}}'_{n-1} + C_4\, \bar{\bar{p}}\, \tilde{\tilde{x}}''_{n-1}, \\ \tilde{\tilde{x}}''_n = p\, \tilde{\tilde{x}}_{n-1} + \bar{p}\, \tilde{\tilde{x}}'_{n-1} + \bar{\bar{p}}\, \tilde{\tilde{x}}''_{n-1} \end{cases}$$

for $n = 1,2,\ldots$, where $\tilde{\tilde{x}}_1 = \tilde{x}_1$, $\tilde{\tilde{x}}'_1 = \tilde{x}'_1$, $\tilde{\tilde{x}}''_1 = \tilde{x}''_1$. It follows from (34) and (35) that

$$(36)\qquad \begin{cases} 0 \leqslant \tilde{x}_i \leqslant \tilde{\tilde{x}}_i, \\ 0 \leqslant \tilde{x}'_i \leqslant \tilde{\tilde{x}}'_i, \\ 0 \leqslant \tilde{x}''_i \leqslant \tilde{\tilde{x}}''_i \end{cases}$$

for $i = 1,2,\ldots$

We want to obtain convergence to zero for the recurrence sequence (35). A sufficient condition for this sequence to be convergent to zero is (see [5] p. 208 formula (8.3.6)) that eigenvalues λ_1, λ_2 and λ_3 of the matrix

$$(37)\qquad \begin{Vmatrix} C_3\, p, & C_3\, \bar{p}, & C_3\, \bar{\bar{p}} \\ C_4\, p, & C_4\, \bar{p}, & C_4\, \bar{\bar{p}} \\ p, & \bar{p}, & \bar{\bar{p}} \end{Vmatrix}$$

fulfil the inequality

$$(38)\qquad |\lambda_i| < 1, \quad i = 1,2,3.$$

Solving the characteristic equation for the matrix (37) we obtain

$$(39) \qquad \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = C_3\, p + C_4\, \bar{p} + \bar{\bar{p}}.$$

Using (19) and (21) we have

$$(40) \qquad \lambda_3 = (d^2 + C_1 d + C_2)\, p + (d + C_1)\, \bar{p} + \bar{\bar{p}}.$$

Condition (23) guarantees the fulfilment of inequality (38). Continuing this argumentation (comp. [5] p. 208) we infer that the series

$$\sum_{i=1}^{\infty} \tilde{\tilde{x}}_i, \quad \sum_{i=1}^{\infty} \tilde{\tilde{x}}'_i \quad \text{and} \quad \sum_{i=1}^{\infty} \tilde{\tilde{x}}''_i$$

are convergent, and after applying inequality (37) and the comparative criterion for convergence we conclude that the series

$$(41) \qquad \sum_{i=1}^{\infty} \tilde{x}_i, \quad \sum_{i=1}^{\infty} \tilde{x}'_i \quad \text{and} \quad \sum_{i=1}^{\infty} \tilde{x}''_i$$

are also convergent. Applying the comparative criterion once more and using (33) we obtain the convergence of the norm series

$$\sum_{n=1}^{\infty} \| x_n(t) - x_{n-1}(t) \|, \quad \sum_{n=1}^{\infty} \| x'_n(t) - x'_{n-1}(t) \| \quad \text{and}$$

$$\sum_{n=1}^{\infty} \| x''_n(t) - x''_{n-1}(t) \|$$

in D. Hence, by the Weierstrass criterion, we obtain uniform convergence of the series

$$\sum_{n=1}^{\infty} [x_n - x_{n-1}], \quad \sum_{n=1}^{\infty} [x'_n - x'_{n-1}] \quad \text{and} \quad \sum_{n=1}^{\infty} [x''_n - x''_{n-1}]$$

in this interval. It follows from this that the series

$$x_0 + \sum_{n=1}^{\infty}[x_n - x_{n-1}] = x$$

is uniformly convergent in interval D. By virtue of this statement the sequence $\{x_n\}$ is uniformly convergent and therefore the function

$$x = \lim_{n\to\infty} x_n$$

is continuous in D.

Vectors (26) and (27) satisfy the assumptions of Lemmas 4 and 5. Thus, the following inequalities hold

$$\| A[x_n] - A[x_{n-1}]\| \leq c_1\ Q[x_n, x_{n-1}]$$

and

$$\| B[x_n] - B[x_{n-1}]\| \leq c_2\ Q[x_n, x_{n-1}].$$

From the definition (8) we get

$$\| A[x_n] - A[x_{n-1}]\| \leq c_1\ (p\ \tilde{x}_n + \bar{p}\ \tilde{x}'_n + \bar{\bar{p}}\ \tilde{x}''_n),$$

$$\| B[x_n] - B[x_{n-1}]\| \leq c_2\ (p\ \tilde{x}_n + \bar{p}\ \tilde{x}'_n + \bar{\bar{p}}\ \tilde{x}''_n),$$

and the convergence of coordinates of the vector sequences $A[x_n]$ and $B[x_n]$, $n = 1,2,\ldots$, to the suitable coordinates of some limit vectors A and B follows from the convergence of series (41). Since all the functional sequences with terms appearing in equation (25) are uniformly convergent in D, we may pass with limit under the integral sign in equations (25) - (29). Then we obtain

$$(42)\qquad x(t) = \int_{t_1}^{t}\Big\{\int_{t_1}^{s} f[x](u)du\Big\}ds + A(t - t_1) + B, \qquad t \in D,$$

where vectors $A, B \in R^m$ are defined by formulae

$$(43)\qquad \begin{aligned} A = \frac{1}{\Delta}\Big\{&\alpha_0(r_2 - \beta_2 f[x](t_2)) - \beta_0(r_1 - \alpha_2 f[x](t_1)) + \\ &- \alpha_0\beta_0\int_{t_1}^{t_2}\Big(\int_{t_1}^{s} f[x](u)du\Big)ds - \alpha_0\alpha_1\int_{t_1}^{t_2} f[x](u)du\Big\}, \end{aligned}$$

$$B = \frac{1}{\Delta}\Big\{\beta_0(t_2 - t_1)(r_1 - \alpha_2\, f[x](t_1)) +$$

(44)
$$+ \alpha_1\big(r_1 - \beta_2\, f[x](t_1)\big) - \alpha_1\big(r_2 - \beta_2\, f[x](t_2)\big) +$$

$$+ \alpha_1\beta_0 \int_{t_1}^{t_2} \Big(\int_{t_1}^{s} f[x](u)du\Big) ds + \alpha_1\beta_1 \int_{t_1}^{t_2} f[x](u)du\Big\} .$$

The system of equations (42) - (44) is equivalent to the system (1), (4). The function x from the system (42) - (44) satisfies the boundary condition (4) because the pointwise convergence of the functional sequence $\{x_n\}$ follows from its uniform convergence, in particular at the points t_i, $i = 1,2$. This means that the function defined by formula (42) is a solution of boundary value problem (1), (4).

Now we shall prove the uniqueness of this solution. Suppose the contrary, i.e. that there exist two different solutions x and $\bar{x}$ together with the corresponding vectors A, B, $\bar{A}$, $\bar{B}$ to the problem (1), (4), and that these solutions are both of the class C^2 in D. Functions x and $\bar{x}$ must fulfil equation (42). Thus, they satisfy the assumptions of Lemmas 6, 7 and 8. Hence we obtain the inequalities

(45)
$$|||\bar{x} - x||| \le C_3\, Q[\bar{x},x],$$

(46)
$$|||\bar{x}' - x'||| \le C_4\, Q[\bar{x},x]$$

and

(47)
$$|||\bar{x}'' - x''||| \le Q[\bar{x},x].$$

Using (8) we have

(48)
$$p\, \tilde{x} \le p\, C_3\, (p\, \tilde{x} + \bar{p}\, \tilde{x}' + \bar{\bar{p}}\, \tilde{x}''),$$

(49)
$$\bar{p}\, \tilde{x}' \le \bar{p}\, C_4\, (p\, \tilde{x} + \bar{p}\, \tilde{x}' + \bar{\bar{p}}\, \tilde{x}''),$$

(50)
$$\bar{\bar{p}}\, \tilde{x}'' \le \bar{\bar{p}}\, (p\, \tilde{x} + \bar{p}\, \tilde{x}' + \bar{\bar{p}}\, \tilde{x}'').$$

Adding side to side inequalities (48) - (50) we obtain

(51) $$(p\,\tilde{x} + \bar{p}\,\tilde{x}' + \bar{\bar{p}}\,\tilde{x}'')\left[1 - (C_3\,p + C_4\,\bar{p} + \bar{\bar{p}})\right] \leqslant 0$$

with

$$\begin{cases} \tilde{x} = |||\bar{x} - x||| \,, \\ \tilde{x}' = |||\bar{x}' - x'|||, \\ \tilde{x}'' = |||\bar{x}'' - x''|||. \end{cases}$$

In view of (23), inequality (51) is fulfilled only for $\tilde{x} = \tilde{x}' = \tilde{x}'' = 0$, which means that solutions $\bar{x}$ and x are the same.

The estimations (24) follow from the formula (8.3.8) of the mentioned handbook [6].

This completes the proof of our theorem.

The existence and uniqueness theorem may be obtained as a corollary to Theorem 1 by taking $k = 1$ and $w_1(t) = t$. The analoguous theorem may also be obtained under weaker assumptions.

T h e o r e m 2. Let a continuous function f of the type

$$f : D \times R^m \times R^m \longrightarrow R^m$$

fulfil the Lipschitz condition

$$\|f(t,x,y) - f(t,\bar{x},\bar{y})\| \leqslant p(t)\,\|x - \bar{x}\| + \bar{p}(t)\,\|y - \bar{y}\|\,, \quad t \in D,$$

where p and $\bar{p}$ are non-negative continuous real functions defined in interval D and satisfying there the inequality

$$|||(d^2 + C_1 d + C_2)\,p + (d + C_1)\,\bar{p}\,||| < 1,$$

where in constants C_1 and C_2 are defined by formulae (15) and (17). Then, under assumption (5) in interval D there exists a unique solution of the class C^2 of the system of ordinary differential equations

$$x''(t) = f\big(t, x(t), x'(t)\big)$$

satisfying boundary value condition

$$\begin{cases} \alpha_0 x(t_1) + \alpha_1 x'(t_1) = r_1 \\ \beta_0 x(t_2) + \beta_1 x'(t_2) = r_2. \end{cases}$$

The sequence of succesive approximations is defined by the formulae

$$x_n(t) = \int_{t_1}^{t_2} \left\{ \int_{t_1}^{s} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du \right\} ds + A[x_{n-1}](t - t_1) +$$

$$+ B[x_{n-1}]$$

where the vectors $A[x_{n-1}]$, $B[x_{n-1}] \in R^m$ are defined in the following manner

$$A[x_{n-1}] = \frac{1}{\Delta} \left\{ \alpha_0 r_2 - \beta_0 r_1 - \alpha_0 \beta_0 \int_{t_1}^{t_2} \left(\int_{t_1}^{s} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du \right) ds + \right.$$

$$\left. - \alpha_0 \alpha_1 \int_{t_1}^{t_2} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du \right\},$$

$$B[x_{n-1}] = \frac{1}{\Delta} \left\{ \beta_0 (t_2 - t_1) r_2 - \beta_0 \alpha_1 \int_{t_1}^{t_2} \left(\int_{t_1}^{s} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du \right) ds + \right.$$

$$\left. + \beta_1 r_1 - \alpha_1 r_2 + \alpha_1 \beta_1 \int_{t_1}^{t_2} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du \right\},$$

for $n = 1, 2, \ldots$, whereas

$$x'(t) = \int_{t_1}^{t} f\left(u, x_{n-1}(u), x'_{n-1}(u)\right) du + A[x_{n-1}], \quad t \in D.$$

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