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INVARIANT SUBMANIFOLDS OF AN $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-MANIFOLD

Invariant submanifolds of almost complex, almost contact and f-structure manifolds have been studied by Schouten and Yano [2]; Yano and Ishihara [4]; and Vohra and Singh [3] respectively. In this paper, we study invariant submanifolds of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold. We mainly define and study two particular invariant submanifolds, which we call horizontal transversal submanifold and vertical transversal submanifold. We prove that an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold does not admit any horizontal transversal submanifold while each vertical transversal submanifold is an almost quaternion manifold with Hermitian metric. Finally, we prove that if the structure tensors $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ in $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold are covariantly constant, then each vertical transversal submanifold of it is necessarily a minimal submanifold.

1. Introduction

Let V^m be an m-dimensional C^∞ Riemannian manifold imbedded differentiably as a submanifold in an n-dimensional C^∞ Riemannian manifold M^n $(m < n)$ by an imbedding map $\Phi : V^m \to M^n$. Let B be the Jacobian map of Φ i.e. $B : T(V^m) \to T(M^n)$ where $T(V^m)$ and $T(M^n)$ are the tangent bundles of V^m and M^n respectively. Let T(V,M) and N(V,M) be the set of all those vectors of $T(M^n)$ which are respectively tangential and normal to the submanifold $\Phi(V^m)$. The set N(V,M) forms a vector bundle over $\Phi(V^m)$ which is called the normal bundle of $\Phi(V^m)$ while the vector bundle induced by Φ from N(V,M) denoted by N(V) is called the normal bundle of V^m. The mapping B : $T(V^m) \rightarrow T(V,M)$ is an isomorphism [4] and we denote by C: $N(V) \rightarrow N(V,M)$ the natural isomorphism. Let $C^\infty - \mathcal{J}_s^r(V^m)$ and $C^\infty - \mathcal{N}_s^r(V^m)$ be the spaces of all C^∞ tensor fields of type (r,s) associated with $T(V^m)$ and N(V) respectively. Since B is an isomorphism [4] we have

$$[BX, BY] = B[X,Y] \tag{1.1}$$

for all X, Y $\in C^\infty - \mathcal{J}_0^1(V^m)$.

Let $\tilde{g}$ be the Riemannian metric tensor in M^n, we then define g and g* by

$$g(X,Y) = \tilde{g}(BX,BY)\circ\Phi \tag{1.2}$$

and

$$g^*(N_1,N_2) = \tilde{g}(CN_1,CN_2) \tag{1.3}$$

for all X, Y $\in C^\infty - \mathcal{J}_0^1(V^m)$, and N_1, $N_2 \in C^\infty - \mathcal{N}_0^1(V^m)$. It is easy to verify that g is a Riemannian metric tensor in V^m and we call it the metric tensor of V^m induced by $\tilde{g}$ while the tensor field g* which is an inner product in $N(V^m)$ is called the metric of N(V) induced by $\tilde{g}$.

The Riemannian connection $\tilde{\nabla}$, corresponding to the metric tensor $\tilde{g}$ in M^n, induces a Riemannian connection ∇ in $\Phi(V^m)$ defined by

$$\tilde{\nabla}_{BX} BY = B\nabla_X Y + CK(X,Y), \tag{1.4}$$

where K(X,Y) is the second fundamental tensor of the submanifold $\Phi(V^m)$ defined by

$$CK(X,Y) = \tilde{\nabla}_{BX} BY - B\nabla_X Y, \tag{1.5}$$

which satisfies

$$K(X,Y) = K(Y,X) \tag{1.6}$$

for all X, Y $\in C^\infty - \mathcal{J}_0^1(V^m)$.

The other second fundamental tensor H is defined by

$$(1.7) \qquad H(X,U) = -(\nabla_X C)U = (\tilde{\nabla}_{BX} CU) - C\,\nabla_X U.$$

for $X \in C^\infty - \mathcal{T}^1_0(V^m)$ and $U \in C^\infty - \mathcal{H}^1_0(V^m)$. These two second fundamental tensors are related by

$$(1.8) \qquad g^*(K(X,Y),U) = g(H(X,U),Y).$$

Let $\tilde{R}$ and $\bar{R}$ denote the Riemannian curvature tensor of the enveloping manifold M^n and the submanifold $\Phi(V^m)$ respectively. We then have

$$(1.9) \quad \tilde{R}(BX, BY)\,BZ = B\bar{R}(X,Y)Z - B\big(H(X,K(Y,Z)) - H(Y,K(X,Z))\big) + \\ + C\left((\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z)\right)$$

for any $X, Y, Z \in C^\infty - \mathcal{T}^1_0(V^m)$.

Let $X_1, X_2, \ldots, X_m$ be m local orthonormal vector fields in V^m, where $m = \dim V^m$, then an element A of $\mathcal{H}^1_0(V^m)$ defined by

$$(1.10) \qquad mA = \sum_{i=1}^{m} K(X_i, X_i)$$

is called the mean curvature vector of the submanifold $\Phi(V^m)$ and the submanifold $\Phi(V^m)$ is called a minimal submanifold of M^n if its mean curvature vector vanishes.

2. Invariant submanifold of an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold

Let M^n be a $C^\infty(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold i.e. it admits two non-zero (1,1) tensor fields $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ such that

$$(2.1) \qquad \begin{cases} \tilde{f}^3_{(1)} + \tilde{f}_{(1)} = 0; \quad \tilde{f}^3_{(2)} + \tilde{f}_{(2)} = 0 \\ \tilde{f}^2_{(1)} = \tilde{f}^2_{(2)}; \quad \tilde{f}_{(1)}\,\tilde{f}_{(2)} = -\,\tilde{f}_{(2)}\,\tilde{f}_{(1)}. \end{cases}$$

Rank $(\tilde{f}_{(1)})$ is a constant integer all over M^n. If we put

$$\tilde{f}_{(3)} = \tilde{f}_{(1)}\,\tilde{f}_{(2)} = -\,\tilde{f}_{(2)}\,\tilde{f}_{(1)}$$

then

$$\begin{cases} \tilde{f}^3_{(3)} + \tilde{f}_{(3)} = 0; & \tilde{f}_{(2)}\,\tilde{f}_{(3)} = -\,\tilde{f}_{(3)}\,\tilde{f}_{(2)} = \tilde{f}_{(1)}; \\ \tilde{f}_{(2)} = \tilde{f}_{(3)}\,\tilde{f}_{(1)} = -\,\tilde{f}_{(1)}\,\tilde{f}_{(3)}; & \tilde{f}^2_{(1)} = \tilde{f}^2_{(2)} = \tilde{f}^2_{(3)} \end{cases}$$

and

$$\text{Rank}\,(f_{(1)}) = \text{Rank}\,(f_{(2)}) = \text{Rank}\,(f_{(3)}).$$

Corresponding to two complementary projection operators $\tilde{l}$ and $\tilde{m}$ defined by

$$(2.3) \qquad \tilde{l} = -\,\tilde{f}^2_{(a)}, \quad \tilde{m} = 1 + \tilde{f}^2_{(a)} \quad (a=1,2,3),$$

where 1 denotes the unit tensor, there exist two complementary distributions which we call the horizontal distribution and the vertical distribution respectively. We note

$$(2.4) \qquad \begin{aligned} &\tilde{f}_{(a)}\,\tilde{l} = \tilde{l}\,\tilde{f}_{(a)} = \tilde{f}_{(a)}; \quad && \tilde{f}_{(a)}\,\tilde{m} = \tilde{m}\,\tilde{f}_{(a)} = 0 \\ &\tilde{f}^2 l = -\,1; && \tilde{f}^2_{(a)}\,\tilde{m} = \tilde{m}\,\tilde{f}^2_{(a)} = 0. \end{aligned}$$

Thus an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-structure acts as an almost quaternion structure on the horizontal distribution. Consequently the dimension of the horizontal distribution is of the form 4r for some constant integer r.

It is known that $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold always admits a positive definite Riemannian metric $\tilde{g}$ such that

$$(2.5) \qquad \tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{f}_{(a)}\tilde{X},\ \tilde{f}_{(a)}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X},\tilde{Y})$$

for all X, $Y \in C^\infty - \mathcal{J}^1_0\,(M^n)$.

We now define an invariant submanifold of M^n. Let V^m be a C^∞ m-dimensional manifold imbedded as a submanifold in the $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n. The submanifold V^m is defined to be an <u>invariant submanifold</u> of M^n if $m > (n-4r)$ and the linear mappings $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ leave invariant the tangent space $T_p(\Phi(V^m))$ of $\Phi(V^m)$ for each point $p \in \Phi(V^m)$.

In the rest of the paper, we shall consider V^m to be an invariant submanifold of M^n. Thus for an arbitrary $X \in C^\infty-\mathcal{J}_0^1(V^m)$, we have

$$\text{(2.6)} \qquad \begin{cases} \tilde{f}_{(1)}(BX) = B(X_1) \\ \tilde{f}_{(2)}(BX) = B(X_2) \end{cases}$$

for some $X_1, X_2 \in C^\infty-\mathcal{J}_0^1(V^m)$. The fact that both X_1 and X_2 are uniquely determined by X enables us to define two (1,1) tensor fields $f_{(1)}$ and $f_{(2)}$ in V^m by

$$\text{(2.7)} \qquad \begin{cases} f_{(1)}X = X_1 \\ f_{(2)}X = X_2. \end{cases}$$

Now from (2.6) and (2.7), we have

$$\text{(2.8)} \qquad \tilde{f}_{(1)}(BX) = B(f_{(1)}X)$$

and

$$\text{(2.9)} \qquad \tilde{f}_{(2)}(BX) = B(f_{(2)}X).$$

We now prove the following theorem.

T h e o r e m 2.1. An invariant submanifold V^m of an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n admits an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-structure induced by the $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-structure.

P r o o f . Operating $\tilde{f}^2_{(1)}$ on (2.8), $\tilde{f}^2_{(2)}$ on (2.9), and in view of (2.1) we have

$$f^3_{(1)} + f_{(1)} = 0; \quad f^3_{(2)} + f_{(2)} = 0$$

while the relations (2.1), (2.8) and (2.9) yield

$$f_{(1)} f_{(2)} = - f_{(2)} f_{(1)} \quad \text{and} \quad f^2_{(1)} = f^2_{(2)}.$$

Now the proof is completed if $f_{(1)}$ and $f_{(2)}$ are shown to be non-zero.

Suppose that $f_{(1)} = 0$, i.e. $f_{(1)}X = 0$ for $X \in C^\infty - \mathcal{T}^1_0(V^m)$. Hence by (2.8) $\tilde{f}_{(1)}(BX) = 0$ and $T(\Phi(V^m)) \subset$ the null space of $\tilde{f}_{(1)}$. Then $\dim V^m = m \leq n - 4r$, since rank $(\tilde{f}_{(1)}) = 4r$. But this contradicts the assumption that $m > n - 4r$. Hence $f_{(1)} \neq 0$. Similarly it can be proved that $f_{(2)} \neq 0$.

T h e o r e m 2.2. The Nijenhuis tensors corresponding to the (1,1) tensor fields appearing in the $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-structure and the induced $(f_{(1)}, f_{(2)})$-structure are related by

$$\text{(2.10)} \quad [\tilde{f}_{(c)}, \tilde{f}_{(d)}](BX, BY) = B[f_{(c)}, f_{(d)}](X, Y) \qquad (c,d=1,2,3).$$

The proof follows from the definition of Nijenhuis tensor and the relation (1.1).

We now define two special invariant submanifolds of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold as follows.

D e f i n i t i o n 2.2. An invariant horizontal transversal submanifold of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold M^n is an invariant submanifold of M^n whose tangent space $T_p(\Phi(V^m))$ for each point $p \in \Phi(V^m)$ does not contain any non-zero element of the horizontal distribution.

D e f i n i t i o n 2.3. An invariant vertical transversal submanifold of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold is an invariant submanifold V^m of M^n whose tangent space $T_p(\Phi(V^m))$ for each point $p \in \Phi(V^m)$ does not contain any non-zero element of the vertical distribution.

We now prove the following theorem.

T h e o r e m 2.3. An $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold does not admit any invariant horizontal transversal submanifold.

P r o o f . Suppose that V^m is an invariant horizontal transversal submanifold of M^n. Consequently, non-zero vector fields of type BX for $X \in C^\infty - \mathcal{T}_0^1(V^m)$ do not belong to the horizontal distribution.

Since by (2.3)

$$\tilde{l}(BX) = - \tilde{f}^2_{(a)}(BX) = B(-f^2_{(a)}X); \qquad (a=1,2,3)$$

and $\tilde{l}(BX)$ is a vector field in the horizontal distribution, therefore $\tilde{l}(BX)$ cannot be a non-zero vector field. Consequently

$$(2.11) \qquad \tilde{l}(BX) = 0$$

i.e. BX belongs to the vertical distribution.

Operating $\tilde{f}_{(a)}$ on the equation (2.11), we get $\tilde{f}_{(a)}(BX) = 0$ which implies that $\dim V^m = m \leq n-4r$. This contradicts the assumption $m > n-4r$ appearing in the definition of the invariant submanifold.

As a corollary of the above theorem, we can state

C o r o l l a r y 1.1. Every invariant submanifold of an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold contains at least one non-zero element of the horizontal diatribution.

Next, let V^m be an invariant vertical transversal submanifold of $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold. Thus non-zero vector fields of type BX for $X \in C^\infty - \mathcal{T}_0^1(V^m)$ do not belong to the vertical distribution.

Since by (2.3)

$$\tilde{m}(BX) = (1 + \tilde{f}^2_{(a)})(BX) = B(X + f^2_{(a)}X) \qquad (a=1,2,3)$$

and $\tilde{m}(BX)$ belongs to the vertical distribution, therefore $\tilde{m}(BX)$ cannot be a non-zero vector field. Consequently $\tilde{m}(BX)=0$ i.e. BX belongs to the horizontal distribution and as a consequence $\dim V^m = m \leq 4r$.

Since

$$0 = \tilde{m}(BX) = (1 + \tilde{f}^2_{(a)})(BX) = B(X + f^2_{(a)}X)$$

and B is an isomorphism, we have

$$f^2_{(a)} = -1 \qquad (a=1,2,3).$$

Similarly

$$f_{(a)}f_{(b)} = - f_{(b)}f_{(a)} \qquad (a\neq b,\ a,b=1,2,3).$$

Thus an invariant vertical transversal submanifold V^m admits an almost quaternion structure [6] induced by the $(f_{(1)},f_{(2)})$--structure. As a consequence, the dimension of such a manifold is $4q \leqslant 4r$, q being a constant integer. In view of the relations (1.3), (2.5), (2.8) and (2.9) we have

$$g(f_{(a)}X,\ f_{(a)}Y) = g(X,Y) \qquad (a=1,2,3).$$

Combining the above facts, we have the following theorem.

T h e o r e m 2.4. An invariant vertical transversal submanifold V^m of an $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n admits an induced almost quaternion $(f_{(1)},f_{(2)},f_{(3)})$-structure as well as an induced Hermitian metric g. The dimension of such a manifold is $4q \leqslant 4r$.

We now assume that the (1,1) tensor fields $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ are covariantly constant with respect to the Riemannian connection $\tilde{\nabla}$. Then in view of (1.4), we have

$$\tilde{\nabla}_{BX}(B\ f_{(a)}Y) = B\ \nabla_{BX}(f_{(a)}Y) + CK(X,f_{(a)}Y) \qquad (a=1,2,3)$$

or

$$\tilde{\nabla}_{BX}(\tilde{f}_{(a)}BY) = B\Big((\nabla_{BX}f_{(a)})(Y) + f_{(a)}(\nabla_{BX}Y)\Big) + CK(X,f_{(a)}Y)$$

hence

$$(\tilde{\nabla}_{BX}\ \tilde{f}_{(a)})(BY) + \tilde{f}_{(a)}(\tilde{\nabla}_{BX}\ BY) =$$

$$= B(\nabla_{BX}f_{(a)})(Y) + B(f_{(a)}\nabla_{BX}Y) + CK(X,f_{(a)}Y)$$

or

$$\tilde{f}_{(a)}(B\nabla_X Y + CK(X,Y)) = B(\nabla_X f_{(a)})(Y) + B(f_{(a)}\ \nabla_X Y) + K(X,f_{(a)}Y)$$

or

$$\tilde{f}_{(a)}CK(X,Y) = B(\nabla_X f_{(a)})(Y) + K(X,f_{(a)}Y).$$

Since the left hand side of the above equation is normal to $\Phi(V^m)$ equating tangential and normal parts, we get

(2.12) $\quad \nabla_X f_{(a)} = 0 \quad$ and $\quad \tilde{f}_{(a)} CK(X,Y) = CK(X, f_{(a)}Y)$

which in view of (1.6) and the fact that C is an isomorphism, yields

(2.13) $\quad K(X, f_{(a)}Y) = K(f_{(a)}X, Y) \quad (a=1,2,3).$

Writing (2.13) for a=1, and taking $f_{(2)}X$ and $f_{(3)}X$ in place of X and Y we get

$$K(f_{(2)}X,\ f_{(2)}Y) = -\ K(f_{(3)}X,\ f_{(3)}Y)$$

which in view of the relation (2.13) gives

(2.14) $$K(X,Y) = 0.$$

Consequently the mean curvature vector of an invariant transversal vertical submanifold vanishes. Thus we have the following theorem.

Theorem 2.5. If the structure tensor fields $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-manifold M^n are covariantly constant, then every invariant vertical transversal submanifold of M^n has vanishing second fundamental tensor K and consequently is a minimal submanifold and $f_{(1)}$ and $f_{(2)}$ are covariantly constant.

We next prove a theorem.

Theorem 2.6. If any two of the Nijenhuis tensors $[\tilde{f}_{(a)}, \tilde{f}_{(a)}]$ (a=1,2,3) $[\tilde{f}_{(a)}, \tilde{f}_{(b)}]$ ($a\neq b$, a,b=1,2,3) of an $(\tilde{f}_{(1)}, \tilde{f}_{(2)})$-structure manifold M^n vanish, then in an invariant vertical transversal submanifold of M^n all the Nijenhuis tensors corresponding to the induced almost quaternion $(f_{(1)}, f_{(2)}, f_{(3)})$-structure vanish.

Proof

The proof follows from the relations (2.8) and a theorem 3.9 in [6].

Let $\tilde{R}$ and $\bar{R}$ be the curvature tensors of the connections $\tilde{\nabla}$ and ∇ in $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n and its invariant vertical transversal submanifold V^m respectively. If the structure tensors $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ are covariantly constant, then in view of Theorem 2.5 and the relation (1.9) we get

$$\tilde{R}(BX,\ BY)BZ = B(\bar{R}(X,Y)Z). \tag{2.15}$$

Next we prove the theorem.

T h e o r e m 2.7. If in a locally flat $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n the structure tensor fields $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ are covariantly constant, then the induced almost quaternion $(f_{(1)},f_{(2)},f_{(3)})$-structure of an invariant vertical transversal submanifold of M^n is integrable.

P r o o f . Since $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ are covariantly constant from Theorem 2.5, we have

$$\nabla_X f_{(1)} = 0 \quad \text{and} \quad \nabla_X f_{(2)} = 0.$$

Thus as a consequence of Theorem 3.9 in [6], the Nijenhuis tensors $[f_{(a)},f_{(b)}]$, $(a,b=1,2,3)$ vanish. Since $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold is locally flat, the relation (2.15) yields

$$\bar{R}(X,Y)Z = 0\,.$$

Hence in view of Theorem 6.3 [6], the almost quaternion $(f_{(1)},f_{(2)},f_{(3)})$-structure is integrable.

Let V^m be an invariant submanifold of M^n and $X,Y \in T_p(V^m)$. Let $\mathcal{Y}$ be the two-dimensional plane determined by the vectors X and Y. The sectional curvature of V^m at the point p with respect to the plane $\mathcal{Y}$ is given by

$$'\bar{K}(X,Y) = \frac{\bar{R}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}\,.$$

Let $'\tilde{K}(X,Y)$ be the sectional curvature of M^n at the point p with respect to the plane $\mathcal{Y}$, then in view of (1.9)

$$\tilde{g}\big(\tilde{R}(BX,BY)BX,BY\big) =$$

$$= \bar{R}(X,Y,X,Y) - g\big(H(X,K(Y,X)),Y\big) + g\big(H(Y,K(X,X))\,Y\big)$$

i.e.

(2.16) $$\tilde{R}(BX,BY,BX,BY) =$$

$$= \bar{R}(X,Y,X,Y) - g^*\big(K(X,Y),K(X,Y)\big) + g^*\big(K(Y,Y),K(X,X)\big).$$

Thus

(2.17) $$'\bar{K}(X,Y) = '\tilde{K}(X,Y) + \frac{g^*\big(K(X,Y),K(X,Y)\big) - g^*\big(K(Y,Y),K(X,X)\big)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

which completes the proof of the following theorem.

T h e o r e m 2.8. The sectional curvature $'\bar{K}(X,Y)$ and $'\tilde{K}(X,Y)$ of the invariant submanifold V^m and the enveloping $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold M^n are related by the relation (2.17).

We now assume that the structure tensors $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ of M^n are covariantly constant. Let $\bar{K}_a(X)$ (respectively $\tilde{K}_{(a)}(X)$) denotes the sectional curvature of V^m (respectively M^n) at the point p with respect to the plane determined by the vectors X and $f_{(a)}X$.

Then in view of Theorem 2.5 and the relation (2.17) we have the following theorem.

T h e o r e m 2.9. Let V^m be an invariant vertical transversal submanifold of $(\tilde{f}_{(1)},\tilde{f}_{(2)})$-manifold such that $\tilde{f}_{(1)}$ and $\tilde{f}_{(2)}$ are covariantly constant. Then the sectional curvatures $'\bar{K}_{(a)}(X)$ and $'\tilde{K}_{(a)}X$ of V^m and M^n with respect to the plane determined by X and $f_{(a)}X$, are equal.

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