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ON THE REPRESENTABILITY OF FUNCTIONS IN THE FORM OF A SUM, PRODUCT, AND A PRODUCT OF SUMS

In this paper we shall use algebraic methods to give sufficient and necessary conditions for the representability of a function of n variables in the form of a sum of functions, a product of functions, and a product of sums of functions with disjoint domains. At the same time these conditions will provide effective methods for finding those functions.

Introduction

In practice, one often encounters nomographic equations of the general form

$$F(x_1,\dots,x_n) = \sum_{i=1}^{n} f_i(x_i) = 0 \tag{1}$$

or

$$F(x_1,\dots,x_n) = \prod_{i=1}^{n} f_i(x_i) = 1. \tag{1'}$$

For these equation, one can construct collinear nomograms by means of elementary methods for joining nomograms (see [1]). For equations of the form (1) we use nomograms with parallel scales, and for equations of the form (1') - nomograms with scales on the sides of a triangle or, more general, nomograms of the shape of the letter N. In the case (1) the same aim can be achieved by use of special nomographic rules (see [3]). We also frequently encounter equations of the form

$$(2) \qquad F(x_1,\dots,x_n) \equiv \sum_{j=1}^{n} f_j(x_{j_1},x_{j_2}) + \sum_{j=1}^{n} g_j(x_{j_3}) = 0$$

or

$$(2') \qquad F(x_1,\dots,x_n) \equiv \prod_{j=1}^{n} f_j(x_{j_1},x_{j_2}) \cdot \prod_{j=1}^{n} g_j(x_{j_3}) = 1$$

with dosjoint domains of the functions f_j and g_j.

For equations (1) and (2) with $n \leqslant 9$ one can use nomograms with an oriented transparent. An exhaustive classification of these equations and methods of building nomograms with transparent can be found in a paper of G.S. Chowanski [2]. By taking logarithms of both sides of equations (1′) and (2′) we can reduce them to the forms (1) and (2), respectively.

The above remarks indicate why it is important to represent a function $F(x_1,\dots,x_n)$ in the form of a product or a sum of functions with disjoint domains. This problem is solved in Theorems 1 and 2 of the present work. Both theorems are related by Theorem 3. The form of the function appearing there was suggested by A. Haman [4] and it is as follows

$$G(x_1,\dots,x_n) = \prod_{i=1}^{k} \sum_{\lambda=1}^{l_i} g^i_\lambda(x^i_\lambda) + C,$$

where $l_1 + l_2 + \dots + l_k = n$. Following [4] we shall call this form the second canonical form of nomographic polynomials in n-variables of the n-th order.

Similar problems have been also considered by A. Haman, E. Otto [1] and others. However, the conditions proposed there assume that F is differentiable. In our theorems, we even do not assume that the function F is continuous.

Notation and assumptions of Theorems 1 and 2

Let $F(x_1,\dots,x_n)$ be a real function of n real variables defined in the cube $Y = \bigtimes_{i=1}^{n} X_i$, where $X_i = \{x_i : \alpha_i < x_i < \beta_i\}$ for $i=1,2,\dots,n$.

Let I denote the sequence of indices occurring in the symbols of variables. We divide this sequence into k disjoint and non-empty subsequences I_1, I_2, ... I_k.

For $j = 1,2,\ldots,k$, let $Y_j = \bigtimes_{i\in I_j} X_i$, where the elements of Y_j are denoted by $y_j = (x_i)_{i\in I_j}$. We next distinguish and fix an element $p \in Y$, $p = (a_1, a_2,\ldots,a_n)$. With the aid of this element we build, for each set Y_j $(1 \leq j \leq k)$, the corresponding set $\bar{Y}_j$ as follows

$$\bar{Y}_j = \bigtimes_{i=1}^{n} Z_i, \text{ where } Z_i = \begin{cases} X_i & \text{for } i \in I_j \\ \{a_i\} & \text{for } i \notin I_j. \end{cases}$$

Again using $p \in Y$ we fix, in each set Y_j, an element $p_j \in Y_j$ in the following way

$$p_j = (a_i)_{i \in I_j}.$$

We see from above that for every j there are some mappings between the sets Y_j, $\bar{Y}_j$ and Y. Namely, let $h_j : Y \longrightarrow \bar{Y}_j$ be defined as follows

$$(x_i)_{i \in I} = y \xrightarrow{h_j} \bar{y}_j = (z_i)_{i \in I}$$

where

$$z_i = \begin{cases} x_i & \text{for } i \in I_j \\ a_i & \text{for } i \notin I_j. \end{cases}$$

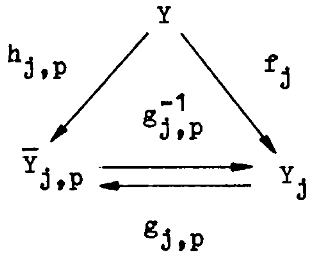
Since $\bar{Y}_j \subset Y$ and $h_j(\bar{y}_j) = \bar{y}_j$, h_j is a special kind of retraction, namely it is the projection of the space Y onto the "hyperspace" $\bar{Y}_j$ passing through $p \in Y$.

Similarly, let $g_j : Y_j \longrightarrow \bar{Y}_j$ be defined as follows

$$(x_i)_{i \in I_j} = y_j \xrightarrow{g_j} \bar{y}_j = (z_i)_{i \in I},$$

where z_i is defined above.

The map g_j is a homeomorphism. The superposition $g_j^{-1} \circ h_j = f_j$ maps the set Y into Y_j. This map is independent of p, because the space Y is a cube. The above relations are illustrated in the diagram



In the formulation of Theorems 1 and 2 we shall use the elements of the cartesian spaces $y \in Y$, $\bar{y}_j \in \bar{Y}_j$ and $y_j \in Y_j$ as the arguments of functions appearing there. Hence we may write y, $h_j(y)$ and $g_j^{-1}\left[h_j(y)\right]$ instead of y, $\bar{y}_j$, y_j, respectively.

The function $F(\bar{y}_j)$ considered in Theorems 1 and 2 can be treated as a restriction of the function F(y) defined on the set Y to the set $\bar{Y}_j \subset Y$. The function $F(\bar{y}_j)$ can also be interpreted as the function $F_j(y)$ being an extension of the function $F(\bar{y}_j)$ defined on the set $\bar{Y}_j$ to the set Y in the following way

$$F_j(y) = \left\{ \begin{matrix} F\left[g_j(y)\right] \text{ for } y \in Y - \bar{Y}_j \\ F(\bar{y}_j) \text{ for } y \in \bar{Y}_j \end{matrix} \right\} = F\left[g_j(y)\right].$$

We can now formulate our theorems.

T h e o r e m 1. In order that a function F(y) defined on Y could be represented in the form

$$\text{(Z1)} \qquad F(y) = \sum_{j=1}^{k} G_j(y_j)$$

it is necessary and sufficient that the following identity hold

$$\text{(T1)} \qquad F(y) = \sum_{j=1}^{k} F(\bar{y}_j) - (k-1)\,F(p).$$

Theorem 2. In order that a function $F(y) \neq 0$ defined on Y could be represented in the form

(Z2) $$F(y) = \prod_{j=1}^{k} G_j(y_j),$$

it is necessary and sufficient that the following identity hold

(T2) $$F(y) = \frac{\prod_{j=1}^{k} F(\bar{y}_j)}{[F(p)]^{k-1}}, \quad \text{where} \quad F(p) \neq 0.$$

Both theorems given above are special cases of the more general Theorem 3. Before we state this theorem we discuss notation and assumptions involved in it.

Similarly as in Theorems 1 and 2 we shall consider a real function $F(x_1,\dots,x_n)$ of n real variables, defined in the cube

$$Y = \bigtimes_{s=1}^{n} X_s, \text{ where } X_s = \{x_s : \alpha_s < x_s < \beta_s\} \text{ for } s = 1, 2,\dots,n.$$

We divide the sequence of indices $I = \{1, 2,\dots,n\}$ into $\sum_{s=1}^{k} m_s$ disjoint non-empty subsequences forming the family

$$U = \{ I_1^1, I_1^2,\dots, I_1^{m_1}, I_2^1, I_2^2,\dots, I_2^{m_2},\dots, I_i^1,\dots, I_i^j,\dots$$
$$\dots, I_i^{m_i},\dots, I_k^1,\dots, I_k^{m_k}\}.$$

We distinguish and fix any element $p \in Y$, $p = (a_s)_{s \in I}$ such that

(3) $$F(p) = F(a_1, a_2,\dots, a_n) \neq 0.$$

For each $I_i^j \in U$ we introduce

1) the subproduct $Y_i^j = \bigtimes_{s \in I_i^j} X_s$ with a fixed element

2) $p_i^j = (a_s)_{s \in I_i^j}$, $p_i^j \in Y_i^j$

and a hyperspace passing through $p \in Y$

$$(4) \quad 3) \qquad \bar{Y}_i^j = \mathop{\times}_{s \in I} Z_s, \text{ where } Z_s = \begin{cases} X_s & \text{if } s \in I_i^j \\ \{a_s\} & \text{if } s \notin I_i^j . \end{cases}$$

Next analogously as before we introduce the maps between the sets Y, $\bar{Y}_i^j$ and Y_i^j.

Theorem 3. In order that a function $F(y)$ defined on Y could be represented in the form

$$(Z3) \qquad F(y) = \prod_{i=1}^{k} \sum_{j=1}^{m_i} G_i^j(y_i^j)$$

it is necessary and sufficient that the following identity hold

$$(T3) \qquad F(y) = \frac{\prod_{i=1}^{k} \left[\sum_{j=1}^{m_i} F(\bar{y}_i^j) - (m_i - 1)\, F(p) \right]}{\left[F(p) \right]^{k-1}} .$$

Proof. First we shall prove the necessity of the above condition. Assume that (Z3) holds. By substituting to (Z3) for each variable $x_s (1 \leq s \leq n)$ the value $x_s = a_s$ being the s-th coordinate of the element p, we obtain

$$(5) \qquad F(p) = \prod_{i=1}^{k} \sum_{j=1}^{m_i} G_i^j(p_i^j).$$

For each fixed i_0 $(i_0 = 1,2,\dots,k)$ and j_0 $(j_0 = 1,\dots,m_{i_0})$ we determine the function $F(\bar{y}_{i_0}^{j_0})$. To this aim we substitute to both sides of identity (Z3) the value $x_s = a_s$ in place of every variable x_s such that $s \notin I_{j_0}^{i_0}$, according to definition (4). Hence we obtain $\sum_{s=1}^{k} m_s$ equations of the form

$$(6)\quad F(\bar{y}_{i_0}^{j_0}) = \frac{\left[\prod_{i=1}^{k} \sum_{j=1}^{m_i} G_i^j(p_i^j)\right] \cdot \sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right) - G_{i_0}^{j_0}\left(p_{i_0}^{j_0}\right) + G_{i_0}^{j_0}\left(y_{i_0}^{j_0}\right)}{\sum_{j=1}^{m_{i_0}} G_{i_0}^j(p_{i_0}^j)} .$$

Clearly from (3) and (5) it follows that $\sum_{j=1}^{m_{i_0}} G_{i_0}^j(p_{i_0}^j) \neq 0$.

From (6) we obtain

$$(7)\quad G_{i_0}^{j_0}\left(y_{i_0}^{j_0}\right) = \frac{F\left(\bar{y}_{i_0}^{j_0}\right)\sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right)}{\prod_{i=1}^{k}\sum_{j=1}^{m_i} G_i^j(p_i^j)} - \sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right) + G_{i_0}^{j_0}\left(p_{i_0}^{j_0}\right).$$

For each fixed i_0 ($i_0 = 1, 2, \ldots, k$) by adding side by side m_{i_0} respective equations of the form (7) we obtain k equations of the form

$$(8)\quad \sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(y_{i_0}^j\right) = \frac{\sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right)}{F(p)} \cdot \sum_{j=1}^{m_{i_0}} F\left(\bar{y}_{i_0}^j\right) + \sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right) - m_{i_0} \cdot \sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right).$$

Transforming (8) we get

$$(9)\quad F(p)\,\frac{\sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(y_{i_0}^j\right)}{\sum_{j=1}^{m_{i_0}} G_{i_0}^j\left(p_{i_0}^j\right)} = \sum_{j=1}^{m_{i_0}} F\left(\bar{y}_{i_0}^{j_0}\right) - (m_{i_0} - 1)\, F(p).$$

Next multiplying side by side k equations of the form (9) we obtain

$$(10)\quad \left[F(p)\right]^k \frac{\prod_{i=1}^{k}\sum_{j=1}^{m_i} G_i^j(y_i^j)}{\prod_{i=1}^{k}\sum_{j=1}^{m_i} G_i^j(p_i^j)} = \prod_{i=1}^{k}\left[\sum_{j=1}^{m_i} F(\bar{y}_i^j) - (m_i-1)\, F(p)\right].$$

Finally making use of (Z3) and (5) we obtain from (10) the thesis (T3) of our theorem.

The sufficiency of condition (T3) is obvious. In fact, introducing the notation for every i ($1 \le i \le k-1$)

$$\bigwedge_{1\le j\le m_{i-1}} G_i^j(y_i^j) = F(\bar{y}_i^j) \quad \text{and} \quad G_i^{m_i}\, y_i^{m_i} = F\left(\bar{y}_i^{m_i}\right) - (m_i-1)\, F(p),$$

and for $i = k$

$$\bigwedge_{1\le j\le m_{k-1}} G_k^j(y_k^j) = \frac{1}{\left[F(p)\right]^{k-1}}\, F(\bar{y}_k^j)$$

and

$$G_k^{m_k}\left(y_k^{m_k}\right) = \frac{1}{\left[F(p)\right]^{k-1}} \cdot \left[F\left(\bar{y}_k^{m_k}\right) - (m_k-1)\, F(p)\right]$$

we obtain (T3). Hence Theorem 3 has been proved.

The identity (Z3) appearing in Theorem 3 takes the form (Z2) in Theorem 2 when for every i ($1 \le i \le k$) we put $m = 1$. Similarly, when $k = 1$ the identity (Z3) takes the form (Z1). Hence Theorems 1 and 2 are special cases of Theorem 3.

E x a m p l e s

1. Let

$F(x,y,z) = \sin(x+y+z) + 4\sin\frac{x+y}{2}\sin\frac{x+z}{2}\sin\frac{y+z}{2}$.

We ask whether this function can be represented in the form

$$F(x,y,z) = G_1(x) + G_2(y) + G_3(z).$$

Let $p = (\pi,\pi,\pi)$. Hence

$F(p) = F(\pi,\pi,\pi) = 0$, $F(\bar{y}_1) = F(x,\pi,\pi) = \sin x$,

$F(\bar{y}_2) = F(\pi, y, \pi) = \sin y$, $F(\bar{y}_3) = F(\pi,\pi, z) = \sin z$.

From above it follows that

$$F(x,y,z) = \sin x + \sin y + \sin z$$

provided that the following identity holds

$$\sin(x+y+z) + 4\sin\frac{x+y}{2}\sin\frac{x+z}{2}\sin\frac{y+z}{2} = \sin x + \sin y + \sin z.$$

which is really the case.

2. Let

(11)
$$F(x_1, x_2, x_3, \dots, x_8) = x_1^2x_4x_5x_6 + x_1^2x_7 + 2x_3x_4x_5x_6 + x_1^2x_8^2 + 2x_1^2x_7x_8 + x_2x_4x_5x_6 + x_2x_7^2 + x_2x_8^2 + 2x_2x_7x_8 + 2x_3x_7^2 + 2x_3x_8^2 - 4x_3x_7x_8 + 2x_4x_5x_6 + 2x_7^2 + 2x_8^2 - x_1^2 - 4x_7x_8 - x_2 - 2x_3 - 2.$$

We ask whether this function can be represented in the form

$$F(x_1,\dots,x_8) = \left[G_1^1(x_1) + G_1^2(x_2) + G_1^3(x_3)\right] \cdot \left[G_2^1(x_4, x_5, x_6) + G_2^2(x_7,x_8)\right].$$

Let $p = (0,\dots,0)$. Hence we have

$$F(p) = F(0,\dots,0) = -2,\ F(\bar{y}_1^1) = F(x_1, 0,\dots,0) = -x_1^2 - 2,$$

$$F(\bar{y}_1^2) = F(0, x_2, 0,\dots,0) = -x_2 - 2$$

$$F(\bar{y}_1^3) = F(0, 0, x_3, 0,\dots,0) = -2x_3 - 2$$

$$F(\bar{y}_2^1) = F(0, 0, 0, x_4, x_5, x_6, 0, 0) = 2x_4x_5x_6 - 2$$

$$F(\bar{y}_2^2) = F(0,\dots,0, x_7, x_8) = 2x_7^2 + 2x_8^2 - 4x_7x_8 - 2.$$

From Theorem 3 it follows that

(12)
$$F(x_1,x_2,\dots,x_8) = \frac{1}{(-2)}\left[(-x_1^2 - 2) + (-x_2 - 2) + (-2x_3 - 2) - 2.(-2)\right] \cdot \left[(2x_4x_5x_6 - 2) + (2x_7^2 + 2x_8^2 - 4x_7x_8 - 2) - 1\cdot(-2)\right] = \left[x_1^2 + x_2 + 2x_3 + 2\right]\cdot\left[x_4x_5x_6 + (x_7 - x_8)^2 - 1\right].$$

provided that the function (11) is identically equal to (12). It is evident that this is true.

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