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Z-AGGREGABLE SETS OF FUNCTIONS

Introduction

The notions of a Z-computable function and a Z-computable set of functions have been introduced in [1]. The notions of a Z-aggregable set of functions and a $*$-connected Z-computable set of functions are introduced in this paper. The aim of this paper is to show that:

1. a Z-aggregable set is a sum of equivalence classes and that these equivalence classes are Z-computable sets;

2. a Z-aggregable set and a Z-computable set are sums of equivalence classes and that these equivalence classes are $*$-connected Z-computable sets.

The notion of a base of the Z-aggregable set is introduced in this paper, too. The notion of $*$-connected, Z-computable set plays the most fundamental role in the construction of the basis for a Z-aggregable set.

1. Basic notions and definitions

Let $\mathcal{R}$ be the set of all real numbers, and $\mathcal{R}^+$ - the set of all non-negative real numbers. By (t,x) we denote the point $(t,x_2,\ldots,x_n)\in\mathcal{R}^n$ if $n\geqslant 2$, and the point $(t)\in\mathcal{R}$ if $n=1$. Let $\Omega_n=\mathcal{R}^+\times\mathcal{R}^{n-1}$. For any set $Z\subseteq\Omega_n$ and for any $a\in\mathcal{R}^+$ we denote $Z_a=\left\{(t,x)\in\Omega_n : (t-a,x)\in Z\right\}$. Let X be an arbitrary non-empty set, to be fixed in the sequel. By $\mathcal{F}^{(n)}$ we denote the set of all mappings $f:\Omega_n\to X$, and by $\mathcal{F}_Z^{(n)}$ the set of all mappings $f: Z\to X$, where $Z\subseteq\Omega_n$. The restriction of $f\in\mathcal{F}^{(n)}$ to $Z\subseteq\Omega_n$ we denote by $f\,|\,Z$.

By the shift operator we mean the operator $\mathcal{P}^{(n)}$ which assings to every mapping $f: Z_a \to X$ the mapping $f^*: Z \to X$ such that $f^*(t,x) = f(t+a,x)$ for every $(t,x) \in Z$. If $\in \mathcal{F}^{(n)}$, then we put $f_{z_a} = (f \mid Z_a)^*$ and, in particular, $f_a = f_{(\Omega)a}$.

W.Żakowski [1] introduced the notion of a Z-computable function. The most characteristic property of such a function is that a function $f \in \mathcal{F}^{(n)}$ is Z-computable iff

$$\forall\,(a,b \geqslant 0)\left[\left(f_{z_a} = f_{z_b}\right) \Rightarrow \left(f_a = f_b\right)\right].$$

In [1] a set $F \subseteq \mathcal{F}^{(n)}$ is said to be Z-injective iff

$$\forall\,(f,g \in F)\left[(f_z = g_z) \Rightarrow (f = g)\right],$$

and a set F is said to be closed under the operation of shift iff

$$\forall\,(f \in F)\,\forall\,(a \in \mathcal{R}^+)(f_a \in F).$$

In [2] for any $H \subseteq \mathcal{F}^{(n)}$ the set $H^* = \left\{f_a : f \in H,\ a \in \mathcal{R}^+\right\}$ is said to be the $*$-closure of the set H. The set $F \subseteq \mathcal{F}^{(n)}$ is Z-computable iff F is Z-injective and $F^* = F$ ($F^* = F$ iff F is closed under the operation of shift, i.e. if F is the $*$-closed set).

2. Z-aggregable sets

Let $\mathcal{F}^{(n)}_{(z)}$ be the set of all Z-computable functions $f: \Omega_n \to X$.

D e f i n i t i o n 1. Functions $f,g \in \mathcal{F}^{(n)}_{(z)}$ are said to be commonly computable (in symbols $f \sim g$) if there exists a Z-computable set $F \subseteq \mathcal{R}^{(n)}$ such taht $f,g \in F$.

C o r o l l a r y 1. The relation $\sim$ is reflexive and symmetric in the set $\mathcal{F}^{(n)}_{(z)}$.

T h e o r e m 1. For any $f \in \mathcal{F}^{(n)}_{(z)}$ and for any $a \in \mathcal{R}^+$, $f \sim f_a$.

P r o o f. For any $f \in \mathcal{F}^{(n)}_{(z)}$ and for any $a \in \mathcal{R}^+$, $f, f_a \in \{f\}^*$. The set $\{f\}^*$ is Z-computable (see Corollary 9 in [2]).

D e f i n i t i o n 2. The set $F \subseteq \mathcal{F}^{(n)}_{(z)}$ is said to be Z-aggregable iff the relation $\sim$ is transitive in the set F and $F^* = F$.

C o r o l l a r y 2. For any Z-aggregable set F the relation $\sim$ in an equivalence relation in the set F.

T h e o r e m 2. For any Z-aggregable set F and for any $f \in F$ the equivalence class $[f]_\sim$ is a Z-computable set.

P r o o f. The set $[f]_\sim$ is $*$-closed, because if $g \in [f]_\sim$ and $h \in \{g\}^*$, then $g \sim h$, hence $h \in [f]_\sim$. The set $[f]_\sim$ is Z-injective, because for any $g, h \in [f]_\sim$, $g \sim h$, hence $\{g,h\}$ is Z-injective. Hence $[f]_\sim$ is Z-computable.

3. $*$-connected Z-computable sets

D e f i n i t i o n 3. For any $f, g \in \mathcal{F}^{(n)}_{(z)}$ we define the relation $\overset{*}{=}$ as follows

$$(f \overset{*}{=} g) \Leftrightarrow \exists (a, b \in \mathcal{R}^+)(f_a = g_b).$$

T h e o r e m 3. The relation $\overset{*}{=}$ is an equivalence relation in the set $\mathcal{F}^{(n)}_{(z)}$.

P r o o f. It is evident that the relation $\overset{*}{=}$ is reflexive and symmetric in the set $\mathcal{F}^{(n)}_{(z)}$. If $f \overset{*}{=} g$ and $g \overset{*}{=} h$, then $\exists (a,b,c,d \in \mathcal{R}^+)\ (f_a = g_b,\ g_c = h_d)$. We denote $m = \max(b,c)$ and $a' = a+m-b$, $d' = d+m-c$. Then $f_{a'} = g_m$ and $g_m = h_{d'}$, hence $f_{a'} = h_d$, $f \sim h$. Hence the relation $\overset{*}{=}$ is transitive in the set $\mathcal{F}^{(n)}_{(z)}$.

C o r o l l a r y 3. For any Z-aggregable set F the relation $\overset{*}{=}$ is an equivalence relation in the set F.

T h e o r e m 4. For any Z-aggregable set F and for any $f,g \in F$ if $f \overset{*}{=} g$, then $f \sim g$.

P r o o f. If $f \overset{*}{=} g$, then $\exists (a,b \in \mathcal{R}^+)\ (f_a = g_b)$. Since $f \sim f_a$, $g_b \sim g$, and F is the Z-aggregable set, we infer that $f \sim g$.

C o r o l l a r y 4. For any Z-aggregable set F and for any $f \in F$ we have $[f]_{\overset{*}{=}} \subseteq [f]_{\sim}$.

T h e o r e m 5. For any Z-aggregable set F and for any $f \in F$ the equivalence class $[f]_{\overset{*}{=}}$ is a Z-computable set.

P r o o f. The set $[f]_{\overset{*}{=}}$ is Z-injective, because $[f]_{\overset{*}{=}} \subseteq [f]_{\sim}$ and $[f]_{\sim}$ is the Z-injective set. If $g \in [f]_{\overset{*}{=}}$ and $h \in \{g\}^*$, then $h \overset{*}{=} g$, hence $h \in [f]_{\overset{*}{=}}$. The set $[f]_{\overset{*}{=}}$ is $*$-closed. Hence $[f]_{\overset{*}{=}}$ is Z-computable.

D e f i n i t i o n 4. The set $F \subseteq \mathcal{F}^{(n)}_{(z)}$ is called $*$-connected iff

$$\forall (f,g \in F)\ (f \overset{*}{=} g).$$

T h e o r e m 6. Let F be a Z-aggregable set such thet $G \subseteq F$ and $f \in G$. The set G is Z-computable und $*$-connected iff $G^* = G$ and $\{f\}^* \subseteq G \subseteq [f]_{\overset{*}{=}}$.

P r o o f. If the set G is Z-computable and $*$-connected, then $\{f\}^* \subseteq G$ and $G \subseteq [f]_{\overset{*}{=}}$ and $G^* = G$. If $G^* = G$ and $G \subseteq [f]_{\overset{*}{=}}$, then G is Z-computable because $[f]_{\overset{*}{=}}$ is Z-computable. The set G is $*$-connected because $[f]_{\overset{*}{=}}$ is $*$-connected.

C o r o l l a r y 5. For any Z-aggregable set F and for any function $f \in F$ the set $\{f\}^*$ is the minimal set, and the set $[f]_{\overset{*}{=}}$ is the maximal set of all Z-computable and $*$-connected sets G such that $G \subseteq F$ and $f \in G$.

4. Basis of Z-aggregable set

D e f i n i t i o n 5. For any Z-aggregable set F a set $G \subseteq F$ is called a basis of the set F iff $F = \bigcup_{f \in G} [f]_{\overset{*}{=}}$ and

$$\forall (f,g \in G) \left\{(f \neq g) \Rightarrow \left[\sim (f \overset{*}{=} g)\right]\right\}.$$

Let $F/_{\overset{*}{=}}$ be the quotient space of the equivalence relation $\overset{*}{=}$.

T h e o r e m 7. Let F be Z-aggregable set and let $G \subseteq F$. G is a basis of the set F iff the set G has exactly one element in common with every set $[f]_{\overset{*}{=}} \in F/_{\overset{*}{=}}$.

P r o o f. If G has exactly one element in common with every set $[f]_{\overset{*}{=}} \in F/_{\overset{*}{=}}$ then $\bigcup_{f \in G} [f]_{\overset{*}{=}} = F$, and if $f,g \in G$ and $f \neq g$, then $f \in [\varphi]_{\overset{*}{=}}$, $g \in [\psi]_{\overset{*}{=}}$ and $[\varphi]_{\overset{*}{=}} \neq [\psi]_{\overset{*}{=}}$, hence $\sim (f \overset{*}{=} g)$.

If $\bigcup_{f \in G} [f]_{\overset{*}{=}} = F$ and $\forall (f,g \in G) \left\{(f \neq g) \Rightarrow \left[\sim (f \overset{*}{=} g)\right]\right\}$, then for every $[f]_{\overset{*}{=}} \in F/_{\overset{*}{=}}$ there exists and element $g \in G \cap [f]_{\overset{*}{=}}$. If g,h are different elements of the set G, then $\sim (g \overset{*}{=} h)$. Hence for every $[f]_{\overset{*}{=}} \in F/_{\overset{*}{=}}$ there exists exactly one element $g \in G \cap [f]_{\overset{*}{=}}$.

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