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THE THEOREM OF DESARGUES IN THE SPACE P^n

The well-known theorem of Desargues (see [1]) concerns perspective triples of points. In this paper we formulate an analogous theorem about triples of k-dimensional hyperplanes in the projective space $P^n (0 \leqslant k < n)$. We shall treat hyperplanes as subsets of the set of points of P^n. First we introduce the following notation and abbreviations:

$\neq (a,b,c \ldots)$ means that the elements $a,b,c,\ldots$ are distinct.

A_i^L - an L-dimensional hyperplane, L can omitted in the symbol A_i^L

$Z(A_1, A_2, \ldots, A_j)$ - the join of hyperplanes $A_1, A_2, \ldots, A_j$ (the smallest hyperplane containing $A_1, A_2, \ldots, A_j$).

T h e o r e m 1. Assume that in the space P^n we are given hyperplanes A_i^k, B_i^k, D^{k+1}, M_i^{k+1}, N_i^{k+1}, $C_i^1 = Z(A_i, B_i)$, $i = 1,2,3$, satisfying the following conditions

(1) $$\bigwedge_{i \neq j} A_i \neq A_j \wedge B_i \neq B_j; \quad i,j = 1,2,3$$

(2) $$\bigwedge_{H^{k+1}} (A_1, A_2 \subset H^{k+1} \rightarrow A_3 \not\subset H^{k+1}) \wedge$$
$$\wedge (B_1, B_2 \subset H^{k+1} \rightarrow B_3 \not\subset H^{k+1})$$

(3) $$\bigwedge_{\neq(i,j,m)} Z(A_i, A_j) = M_m \wedge Z(B_i, B_j) = N_m; \quad i,j,m = 1,2,3$$

(4) $$\bigwedge_i M_i \cap N_i \cap D = D_i^m, \quad \text{where} \quad m \geqslant k; \quad i = 1,2,3.$$

Then we have

(*) $$\bigvee_{C_i^{k+1}, F^k} \bigwedge_i \bar{C}_i, F \subset C_i \quad i = 1,2,3.$$

P r o o f. First observe that from the assumptions it follows that

(5) $$\neq (M_1, M_2, M_3) \quad \text{and} \quad \neq (N_1, N_2, N_3).$$

In fact, if we had $M_1 = M_2$, then we would obtain the inclusion $\bigwedge_i A_i \subset M_1$. Next observe that for the proof of the theorem it suffices to consider the cases $k = n-3$ and $k = n-2$ only. In fact, from (2) and (3) it follows that the hyperplanes $Z_1 = Z(A_1, A_2, A_3)$ and $Z_2 = Z(B_1, B_2, B_3)$ have dimension $k+2$. Moreover, it is clear that $D \subset Z_1 \cap Z_2$ (in view of (5), $D_1 = D_2 = D_3$ cannot hold). Hence the dimension of the hyperplane $Z(Z_1, Z_2)$ is not greater than $k+3$. On the other hand, taking $k = n-1$, we obtain a contradiction with (2) which shows that k cannot exceed $n-2$.

From (1) and (3) it follows that if $i \neq j$, then the dimension of the hyperplane $A_i \cap A_j$ equals $k-1$. Let A_m^* and B_m^* denote respectively the intersections $A_i \cap A_j$ and $B_i \cap B_j$, where $i,j,m = 1,2,3$; $\neq (i\ j\ m)$. In the sequel we shall assume that $k \geqslant 1$, since for $k = 0$ the proof is analogous. Suppose that $A_1^* \neq A_2^*$, then, in virtue of (1) and (3), we infer that $A_1^* \neq A_3^* \neq A_2^*$. This implies that the dimension of $Z(A_1, A_2, A_3)$ is equal $k+1$ which contradicts (2). Hence we obtain the equalities

$$A_1^* = A_2^* = A_3^* = A^* \quad \text{and} \quad B_1^* = B_2^* = B_3^* = B^*.$$

First we consider the case

(6) $$k = n - 3.$$

This equality means that the dimension of the hyperplane $Z(A_1, A_2, A_3, B_1, B_2, B_3)$ equals n. There are two possibilities: $A^* \neq B^*$ or $A^* = B^*$.

α) $\underline{A^* \neq B^*}$

1° $A^* \not\subset D$ and $B^* \not\subset D$. In this case the dimension of the hyperplanes $A_i \cap B_i$ is not less than k-1. In fact, from the relations $M_i \cap M_j = A_m$, $N_i \cap N_j = B_m$ for $i,j,m = 1,2,3 \neq (i,j,m)$ it follows that $D_i \cap D_j \subset A_m$ and $D_i \cap D_j \subset B_m$. Moreover, the hyperplane $F = Z(A^*, B^*)$ is contained in all hyperplanes $C_i = Z(A_i, B_i)$. If the dimension of F were equal to k+1, then there would hold the equality $F = C_i$ for all i which contradicts the assumption.

2° $A^* \subset D$ and $B^* \not\subset D$. We can distinguish the following subcases:

a) $\bigwedge_i A_i \subset D$. Then $\bigwedge_i A^* \subset D_i$ and consequently $\bigwedge_i A^* \subset N_i$.

Suppose that the dimension of the hyperplane $N_1 \cap N_2 \cap N_3$ equals k. In view of the relation $\bigwedge_{\neq(i,j,m)} N_i \cap N_j = B_m$ we then obtain $B_1 = B_2 = B_3$. On the other hand from the inclusion $B^* \subset N_1 \cap N_2 \cap N_3$ it follows that

$$\text{(7)} \qquad \dim N_1 \cap N_2 \cap N_3 = k - 1.$$

Taking into account the fact that $\bigwedge_i A^* \subset N_i$ we obtain $A^* = B^*$ which means that under our assumption the case a) cannot hold.

b) $A_1 \subset D \wedge A_2, A_3 \not\subset D$. Then we have $A_1 = D_2 = D_3$, and consequently $N_2 \cap D = N_3 \cap D = A_1 = B_1$. This contradicts the assumption $B^* \not\subset D$.

c) $A_1, A_2 \subset D \wedge A_3 \not\subset D$. This implies, by (4), the relations $N_2 \cap D = A_1 \wedge N_1 \cap D = A_3$ from which it follows that the dimension of the hyperplanes $A_1 \cap B_1$ and $A_2 \cap B_2$ equal k-1. From the relations $N_1 \cap N_2 = B_3$ and $A^* \subset N_1 \cap N_2$ it follows that the dimension of $A_3 \cap B_3$ is not less than k - 1. Taking $F = B_3$ we obtain the thesis of the theorem.

3° $A^{*} \subset D$ and $B^{*} \subset D$.

a) $\bigwedge_i A_i \subset D \wedge B_i \subset D$. Then it is easy to see that

$$\bigwedge_{i \neq j} D_i \neq D_j. \tag{8}$$

On the other hand we have $\bigwedge_i A^{*}, B^{*} \subset D_i$ which implies $\bigwedge_i Z(A^{*}, B^{*}) = D_i$ contrary to (8).

b) $A_1 \subset D \wedge A_2, A_3 \not\subset D$. These conditions imply, by (3) and (4), the equalities $A_1 = B_1 = Z(A^{*}, B^{*}) = D_1 = D_2 = D_3$, and consequently $\bigwedge_i D_1 \subset N_i$, which contradicts (7).

c) $A_1, A_2 \subset D \wedge A_3 \not\subset D$. This implies $A_1 = D_2$ $A_2 = D_1$ and consequently $A^{*} \subset N_1 \cap N_2$. Since $B^{*} \subset N_1 \cap N_2$ as well, we infer that $Z(A^{*}, B^{*}) = B_3$. If $B_1, B_2 \not\subset D$, then $A_3 = B_3$, a contradiction. Hence suppose that e.g. $B_1 \subset D$ and $B_2 \not\subset D$. This implies $B_1 = D_1 = A_2$. Finally observe that the hyperplanes $C_1 = D$, $C_2 = Z(A_2, B_2)$ and $C_3 = Z(A_3, B_3)$ have dimension $k+1$ and satisfy the condition $\bigwedge_i B_3 \subset C_i$.

β) <u>$A^{*} = B^{*}$</u>

1° $A^{*} \not\subset D$. According to the equalities $\bigwedge_i Z(A^{*}, D_i) = M_i = N_i$ we then have

$$\bigwedge_i A_i = B_i. \tag{9}$$

However, this contradicts (6).

2° $A^{*} \subset D$. In this case, if $\bigvee_{i,j} A_i = B_j$, $i,j = 1,2,3$ then

$$A_i, B_j \subset D. \tag{10}$$

In fact, if we had e.g., $A_1 = B_3$ and $A_1, B_3 \not\subset D$, then, since $D \subset Z_1 \cap Z_2$, we would obtain $Z(D, A_1) = Z(D, B_3) = Z(A_1, A_2, A_3) = Z(B_1, B_2, B_3)$, a contradiction with (6). We shall consider the following subcases:

a) $A_1, A_2 \subset D \wedge (B_1 \subset D \wedge B_2, B_3 \not\subset D \vee B_1, B_3 \subset D \vee B_3 \subset D \wedge$

$$\wedge B_1, B_2 \not\subset D \vee B_1, B_2 \subset D).$$

We investigate all the components of this alternative.

a') $A_1, A_2, B_1 \subset D \wedge B_2, B_3 \not\subset D$. This implies $A_1 = B_1$. Taking $F = B_3$ we obtain the thesis of the theorem, since $B_2 \subset Z(B_3, A_2)$.

a") $A_1, A_2, B_1, B_2 \subset D$. Then we have $A_1 = B_1$ and $A_2 = B_2$. Taking the hyperplanes B_3, M_1, M_2 as F, C_2, C_3, respectively, we see that condition (*) holds.

a"') $A_1, A_2, B_1, B_3 \subset D$. This implies $A_2 = B_3$. Taking A_2, D, N_1, M_1 as F, C_1, C_2, C_3, respectively, we obtain the thesis.

a^{IV}) $A_1, A_2, B_3 \subset D \wedge B_1, B_2 \not\subset D$. Then the hyperplanes A_2 and B_3, as well as A_1 and B_3 would have to be identical, contrary to (1).

b) $A_1 \subset D \wedge A_2, A_3 \not\subset D$. This implies $A_1 = D_2 = D_3$, and consequently $A_1 \subset N_2 \wedge A_1 \subset N_3$. Hence the hyperplanes A_1 and B_2 coincide. We may assume that $B_2, B_3 \not\subset D$, because the converse has been discussed previously. In view of (7) we infer that $D_1 \neq A_1$. Since $M_1 \cap N_1 = D_1$, it follows that the dimension of the hyperplane $Z(M_1, N_1)$ is $k+2$ according to (6) we have $M_1 = N_1$. Hence the hyperplanes $Z(A_2, B_2)$ and $Z(A_3, B_3)$ of dimension $k+1$ have a common part, denoted by F, of dimension k. Taking into account that $A^* \subset F$ we infer that the hyperplane $Z(A_1, F)$ has dimension $k+1$ which concludes the proof of this case.

c) $\bigwedge_i A_i, B_i \not\subset D$. By (1) and (10) we infer that the hyperplanes $A_1, A_2, A_3, B_1, B_2, B_3$ are all distinct. It is easy to see that

$$\bigwedge_i M_i \neq N_i, \quad i = 1, 2, 3. \tag{11}$$

In fact, if we had, say $M_1 = N_1$, then the dimension of the hyperplane $Z(M_1, D)$ would equal $k+2$. In view of the inclusion $A_1, B_1 \;\; Z(M_1, D)$ this is a contradiction with (6).

Let T_1^{k+2}, T_2^{k+2}, T_3^{k+2} denote the hyperplanes $Z(M_i,N_i)$, $i = 1,2,3$, respectively. Similarly, let H_m denote the hyperplanes $Z(A_i,B_i) \cap Z(A_j,B_j)$, where $i,j,m = 1,2,3; \neq (i,j,m)$. From the condition $Z(A_i,B_i)$, $Z(A_j,B_j) \subset T_m$, where $i,j,m = 1,2,3 \neq (i,j,m)$, it follows that the dimension of the hyperplanes H_m, $m = 1,2,3$ is not less than k. Suppose that the dimension of the hyperplane H_1 equals $k + 1$. This implies that $Z(A_2,B_2) = Z(A_3,B_3)$, and consequently $M_1 = N_2$. Hence the hyperplanes H_1,H_2,H_3 have dimension k. Suppose that $H_1 \neq H_2$. Then the hyperplane $Z(C_1,C_2)$ (in this case we have $\bigwedge \bar{C}_i = C_i$), of dimension $k + 2$, contains the hyperplanes H_1,H_2 and consequently it contains also their join, i.e. C_3. But this contradicts (6). Hence we have $H_1 = H_2 = H_3$ and the hyperplanes $C_1,C_2,C_3 H_1$ fulfill the thesis of the theorem.

Now we are going to deal with the case

$$k = n - 2. \tag{12}$$

Similarly as in (6), the equality (12) implies that the join of the hyperplanes A_1,A_2,A_3,B_1,B_2,B_3 has dimension n. As previously, we can distinguish two cases: $A^* \neq B^*$ and $A^* = B^*$. The proof for the former runs analogously as in the case $k = n - 3$. Hence we may consider the second case, i.e. $A^* = B^*$. Depending on the situation of A^* with respect to the hyperplane D we have two possibilities:

1° $A^* \not\subset D$. Then similarly as in case (9) we obtain $A_i = B_i$ for all i which yields the thesis.

2° $A^* \subset D$. We distinguish two subcases:

a) $A_1,A_2 \subset D$. In view of the inclusions $A_1 \subset Z(B_1,B_3) \wedge A_2 \subset Z(B_2,B_3)$, taking $B_3 = F$, $C_1 = Z(B_1,B_3)$, $C_2 = Z(B_2,B_3)$, $C_3 = (A_3,B_3)$, we obtain the thesis of theorem.

b) $A_1 \subset D$ $A_2,A_3 \not\subset D$. This implies that $A_1 = B_1$. If the hyperplanes $Z(A_2,B_2)$ have a common part F of dimension k, the theorem holds. If on the other hand, the dimension of F is $k + 1$, then taking as F the intersection $Z(A_2,B_2) \cap D$ we obtain the thesis.

c) $$\bigwedge_i A_i, B_i \not\subset D.$$

We exclude the trivial case

(13) $$\bigwedge_i A_i = B_i.$$

We embed the space P^n into the (n+1)-dimensional space P^{n+1}. Next we consider a hyperplane H^n such that $H \subset P^{n+1} \wedge H \neq P^n \wedge D \subset H$. It is easy to see that the hyperplanes D_1, D_2, D_3 have dimension k and are all distinct (from $D_1 = D_2$ it follows that $M_1 = M_2$). We take three distinct hyperplanes E_1^{k+1}, E_2^{k+1}, E_3^{k+1} satisfying the conditions: $\bigwedge_i D_i \subset E_i \wedge E_i \subset H^n \wedge \wedge E_i \neq D$ $i = 1,2,3$. Let G_m^k denote the intersections $E_i \cap E_j$, $i,j,m = 1,2,3$, $\neq (i,j,m)$. The triples G_1, G_2, G_3 and A_1, A_2, A_3, as well as G_1, G_2, G_3 and B_1, B_2, B_3 satisfy the assumptions (1) and (4), where $k = (n+1)-3$. Hence in view of the case proced previously we infer that there exist hyperplanes F_1^k, F_2^k, $C_{A,1}^{k+1}$, $C_{A,2}^{k+1}$, $C_{A,3}^{k+1}$, $C_{B,1}^{k+1}$, $C_{B,2}^{k+1}$, $C_{B,3}^{k+1}$ satisfying condition (*). Clearly the inclusions $A^* \subset F_1 \wedge A^* \subset F_2$ hold from which we infer that the dimension of the hyperplane $Z(F_1, F_2)$ does not exceed $k+1$. Observe moreover that we have $F_1, F_2 \not\subset H \wedge F_1, F_2 \not\subset P^n$, and $F_1 = F_2$ cannot hold. Hence the dimension $Z(F_1, F_2)$ is not less than $k+1$. Let F denote the intersection $Z(F_1, F_2) \cap P^n$. According to (13) we have, e.g. $A_1 \neq B_1$. Consider the hyperplanes $Z(A_1, B_1) = C_1^{k+1}$, $Z(A_1, G_1) = C_{A,1}^{k+1}$, $Z(B_1, G_1) = C_{B,1}^{k+1}$, $Z(C_1, C_{A,1}) = W_1^{k+2}$ and $Z(C_1, C_{B,1}) = W_2^{k+2}$. The common part of the hyperplanes W_1 and W_2 contains C_1 and G_1, where $G_1 \not\subset C_1$. This implies $W_1 = W_2$. On the other hand from the inclusion $Z(F_1, F_2) \subset W_1$ it follows that $F \subset W_1$, and hence $F \subset W_1 \cap P^n = C_1$. Similarly we can show that if $A_2 \neq B_2$ and $A_3 \neq B_3$, then $F \subset Z(A_2, B_2) \wedge F \subset Z(A_3, B_3)$. Finally if $A_2 = B_3$, then taking $C_2 = Z(A_2, F)$ we obtain the thesis.

T h e o r e m 2. Assume that in the space P^n we are given hyperplanes A_i^k, B_i^k, F^k, C_i^{k+1}, M_i^{k+1}, N_i^{k+1}, $i = 1,2,3$, satisfying the conditions:

$$(2.1)\qquad \bigwedge_{i\neq j} A_i \neq A_j \wedge B_i \neq B_j, \quad i,j = 1,2,3,$$

$$(2.2)\qquad \bigwedge_{H^{k+1}} (A_i, A_2 \subset H^{k+1} \rightarrow A_3 \subset H^{k+1}) \wedge (B_1, B_2 \subset H^{k+1} \rightarrow B_3 \subset H^{k+1}),$$

$$(2.3)\qquad \bigwedge_i A_i, B_i, F \subset C_i, \quad i = 1,2,3,$$

$$(2.4)\qquad \bigwedge_{\neq(i,j,m)} (A_i, A_j \subset M_m) \wedge (B_i, B_j \subset N_m), \quad i,j,m = 1,2,3.$$

Then we have

$$(**)\qquad \bigvee_{D^{k+1}} \bigwedge_i M_i \cap N_i \cap D = D_i^m, \quad m \geqslant k, \ i = 1,2,3.$$

P r o o f. Similarly as in the proof of Theorem 1 it suffices to consider the cases $k = n-3$ and $k = n-2$ and $k \geqslant 1$ only. We also infer that

$$(2.5)\qquad \begin{gathered} \neq (M_1, M_2, M_3) \text{ and } \neq (N_1, N_2, N_3) \quad , \\ \bigwedge_{i\neq j} (A_i \cap A_j = A^*) \wedge (B_i \cap B_j = B^*) \end{gathered}$$

(clearly, the hyperplanes A^* and B^* have dimension k-1).First we are going to deal with the case

$$(2.6)\qquad k = n - 3.$$

By assumption, the hyperplanes $Z_1 = Z(A_1, A_2, A_3)$ and $Z_2 = Z(B_1, B_2, B_3)$ have dimension $k + 2$. Hence the hyperplane $Z_0 = Z_1 \cap Z_3$ has dimension $k+1$, as $Z_1 \neq Z_2$ by (2.6). Clearly, $\bigwedge_i M_i \cap N_i \subset Z_0$ and $M_i, N_i \subset Z(C_j, C_m)$ for i,j,m and 1,2,3 and $\neq (i,j,m)$. This implies that the dimension of $M_i \cap N_i$, i=1,2,3 is not less than k. Thus the theorem holds provided we take Z_0 as D.

Next let

(2.7) $$k = n - 2$$

i.e. the hyperplane $Z(A_1,A_2,A_3,B_1,B_2,B_3)$ has dimension $k+2$. As before we consider two possibilities $A^* \neq B^*$ or $A^* = B^*$.

α) $\underline{A^* \neq B^*}$

1° $A_1 = F = B_1$. Then, by (2.7) we have $C_2 = Z(A_2,B_2) \neq C_3 = Z(A_3,B_3)$ and $M_2 = N_2 = C_3$ $M_3 = N_3 = C_2$. Taking the hyperplane M_1 as D we obtain the thesis (the hyperplanes $A_2 \cap B_2$ and $A_3 \cap B_3$ are contained in M_1 and N_1).

2° $A_1 = F = B_2$. In this case, in view of the conditions $B_3 \subset M_2 \cap N_2$, $A_3 \subset M_1 \cap N_1$, $F \subset M_3 \cap N_3$ and $Z(A_3,B_3,F) = C_3$ we infer that the hyperplane C_3 satisfies condition (**).

3° $A_1 = F \wedge \bigwedge_i B_i \neq F$. Taking into consideration the inclusions $Z(A_2 \cap B_2, A_3 \cap B_3) \subset M_1 \cap N_1$, $B_3 \subset M_2 \cap N_2$, $B_2 \subset M_3 \cap N_3$ and $Z(A_2 \cap B_2, A_3 \cap B_3) \subset Z(B_2,B_3) = N_1$ we see that the hyperplane N_1 satisfies the thesis.

4° $\bigwedge_i A_i \neq F \neq B_i$. Let E_i^{k-1}, $i = 1,2,3$, denote the intersections of the hyperplanes A_i and B_i. It is clear that all the hyperplanes E_1,E_2,E_3 are distinct. Next observe that $\bigwedge_{\neq(i,j,m)} Z(E_i,E_m) \subset M_i \cap N_i$ for $i,j,m = 1,2,3$. Among the pairs M_i,N_i, $i = 1,2,3$, there are at least two pairs such that $M_i \neq N_i$ (the equality $M_1 = N_1$ and $M_2 = N_2$ would imply $C_1 = C_2 = C_3$, and so on). Suppose that $M_1 \neq N_1 \wedge M_2 \neq N_2$. Then the hyperplanes $M_1 \cap N_1$ and $M_2 \cap N_2$ have dimension k, hence $Z(E_2,E_3)$ and $Z(E_1,E_2)$ also have dimension k. This implies that the hyperplane $Z(E_1,E_2,E_3) = D$ has dimension $k+1$.

β) $\underline{A^* = B^*}$.

We exclude the trivial case

(2.8) $$\bigwedge_i A_i = B_i, \quad i = 1,2,3.$$

1° $A_1 = F = B_1$. This implies $M_2 = N_2 = C_3$ and $M_3 = N_3 = C_3$. Clearly, there exists a hyperplane H^k contained in the intersection $M_1 \cap N_1$. It is easy to see that $H \neq F$ and $A^* \subset H$. Putting $D = Z(H,F)$ we obtain the thesis of the theorem.

2° $A_1 = F = B_2$. Then, in view of the conditions $A_3 \subset M_1 \cap \cap N_1$, $B_3 \cap M_2 \cap N_2$ and $F \subset M_3 \cap N_3$, we see that the hyperplane $D = C_3$ satisfies condition (**).

3° $A_1 = F$ and $\bigwedge_i B_i \neq F$. In this case $D = N_1$ satisfies the thesis.

4° $\bigwedge_i A_i \neq F \neq B_i$. Similarly as in the proof of Theorem 1 we embed P^n into P^{n+1}. Let C^{k+1} be any hyperplane satisfying the conditions $C \not\subset P^n \wedge F \subset C$. Next let $\bar{C}^k$ and $\bar{\bar{C}}^k$ be two distinct hyperplanes such that $\bar{C}, \bar{\bar{C}} \subset C \wedge \bar{C}, \bar{\bar{C}} \not\subset P^n \wedge A^* \subset \bar{C} \cap \bar{\bar{C}}$.

We denote the hyperplanes $Z(C,A_i)$, $Z(C,B_i)$ and $Z(C,C_i)$ by E_i^{k+1}, G_i^{k+1} and K_i^{k+2}, respectively. Let L_i^k denote the intersection $E_i \cap G_i$ (The dimension of L_i is k, because $\bigwedge_i E_i, G_i \subset K_i \wedge E_i \neq G_i$). It is not difficult to verify that $\bigwedge_i A^* \subset L_i$ and $\bar{C} \neq L_i \neq \bar{\bar{C}}$. Suppose that $L_1 = L_2$, then in view of the conditions $Z(L_1,\bar{C}) \cap P^n = A_1$ and $Z(L_2,\bar{C}) \cap P^n = A_2$ we obtain a contradiction with (2.1). Hence the hyperplanes L_1, L_2, L_3 are distinct. Let N_m^{k+1} denote the hyperplanes $Z(L_i,L_j)$, $i,j,m = 1,2,3$, $\neq (i,j,m)$. If the hyperplane $Z_o = Z(L_1,L_2,L_3)$ had dimension k+1, then from the previous consideration it follows that are hyperplane $Z(A_1,A_2,A_3)$ would have dimension k+1, a contradiction. Hence we have shown that the triples A_1,A_2,A_3 and L_1,L_2,L_3 as well as B_1,B_2,B_3 and L_1,L_2,L_3 satisfy the assumptions (2.1) – (2.4). From the case $k = n-3$ proved above it follows that there exist hyperplanes $\bar{D}^{k+1}$ and $\bar{\bar{D}}^{k+1}$, respectively for the first and the second triple, satisfying condition (**). If $Z_o = P^n$ did hold, in view of the conditions $\bigwedge_i E_i \cap P^n = A_i \wedge G_i \cap P^n = B_i$ we would obtain $\bigwedge_i A_i = L_i = B_i$, contrary to (2.8). Thus the hyperplane $D = Z_o \cap P^n$ has dimension k+1. Suppose that $\bar{N}_1 \subset P^n$. Hence we have $\bar{N}_1 = D$ which implies $\bar{N}_2, \bar{N}_3 \not\subset P^n$, and consequently

$\bar{N}_2 \cap P^n = L_3 \wedge N_3 \cap P^n = L_2$. On the other hand we know that $\bigwedge_i L_i \subset E_i \wedge E_i \cap P^n = A_i$ which shows that $A_2 = L_2$ and $A_3 = L_3$. Similarly we obtain the relation $B_2 = L_2$ and $B_3 = L_3$. Finally we see that $M_1 \cap N_1 = D$, $A_1 = B_3 \subset M_2 \cap N_2$ and $A_2 = B_2 \subset M_2 \cap N_2$, i.e. the hyperplane D satisfies the thesis of the theorem.

It remains to consider the case $\bigwedge_i \bar{N}_i \not\subset P^n$, $i = 1,2,3$. Let R_i^k, S_i^k denote the intersections $\bar{N}_i \cap M_i$, $\bar{N}_i \cap N_i$, respectively. Evidently we have $\bigwedge_i R_i, S_i \subset D$. However the dimension of the hyperplanes $\bar{N}_i \cap D$ is not greater than k which implies the equalities $\bigwedge_i R_i = S_i$ and the inclusions $\bigwedge_i R_i \subset M_i \cap N_i$. This ends the proof of the theorem.

Taking in Theorems 1 and 2 $n = 2$ or $n = 3$ and $k = 0$ we obtain the known theorems of Desarques.

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