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PARTITIONS AND COUNTING ISOMORPHISM TYPES OF CERTAIN MAXIMAL GRAPHS WITHOUT 1-FACTORS

Introduction

Number theory and combinatorics (including graph theory) are closely related areas of mathematics. This paper gives one more example which confirms this opinion. Namely, it proves that counting isomorphism types of certain maximal graphs (in particular of those without 1-factors) is a problem of the additive theory of numbers. Simply this problem resolves itself into counting certain restricted partitions of some positive integers into odd parts.

In Sections 2 and 3 there are found auxiliary recurrence equations for the numbers of partitions of certain integers into definite number of odd parts. One of these formulae can be used for the effective tabulation of an auxiliary function φ of two variables. Summing up those of its values which form a definite part of a column of the table, we obtain one of the desired numbers of isomorphism types. We have not succeeded in finding a general recursion formula for these numbers without involving the auxiliary function φ (or one of its equivalents φ_k).

Section 1 gives graph-theoretical essentials including Mader's characterization of maximal graphs of a given order and with a given deficiency. Section 2 gives necessary information from the theory of partitions. The extensions of the formal power series method are outlined [8]. Some simplifications are indicated.

1. Graph-theoretical preliminaries

Now we shall quote some results of graph theory together with necessary definitions. Only finite ordinary graphs will be considered. Let V be a finite set disjoint from the set $\mathcal{P}_2(V)$ of its two-element subsets: $V \cap \mathcal{P}_2(V) = \emptyset$. The cardinality $|V|$ of V will be denoted by n. A graph G with the vertex set $V(G) = V$ and the edge set $E(G) = E$ is an abstract unoriented simplicial complex of dimension 1 or 0 (if $E = \emptyset$) with the vertex set V and with the set E of 1-simplexes (called edges of G), where $E \subseteq \mathcal{P}_2(V)$. The graph G is usually represented as an ordered pair $< V;E >$. The number $n = |V|$ of vertices of G is called the order of G, while $|E(G)|$ is called the size of G.

The graph $< V, \mathcal{P}_2(V) >$ with all possible edges is complete and is denoted by K or by K_n (possibly with distinguishing superscript) if its order is n. Two graphs $G_1 = < V_1, E_1 >$ and $G_2 = < V_2, E_2 >$ are disjoint if they are subgraphs of a certain graph and their vertex sets are disjoint, that is

$$(V_1 \cup V_2) \cap \mathcal{P}_2(V_1 \cup V_2) = \emptyset \quad \text{and} \quad V_1 \cap V_2 = \emptyset.$$

The union $G_1 \cup G_2 := < V_1 \cup V_2, E_1 \cup E_2 >$ of disjoint graphs G_1 and G_2 is denoted by $G_1 + G_2$. Similarly, $\sum_i G_i$ denotes the union of mutually disjoint graphs G_i. The symbol $G_1 * G_2$ stands for the join of two disjoint graphs G_1 and G_2, which equals $G_1 + G_2$ together with all possible edges with one end-vertex from V_1 and the other from V_2.

Incidence of a vertex and an edge, isomorphism of graphs (denoted by $\simeq$ or simply by the equality sign $=$), and isomorphism type of a graph are notions borrowed from the general theory of complexes. Two different vertices or two different edges are adjacent if they are simulataneously incident to another simplex (an edge or a vertex, respectively). The degree $d(x,G)$ of a vertex x in G is the number of edges of G incident to x.

A graph $G_1 = \langle V_1, E_1 \rangle$ is a subgraph of G (i.e., $G_1 \subseteq G$) if $V_1 \subseteq V$ and $E_1 \subseteq E$; G_1 is a factor of G if $V_1 = V$ and $E_1 \subseteq E$. A subgraph of G, in which each vertex is of degree 1, is a matching of G. Hence any two edges of a matching are not adjacent. A matching of G is maximal if its size is maximal. A matching of G containing all the vertices is a 1-factor of G. The number of vertices of G which do not belong to a maximal matching of G is the deficiency $a(G)$ of G. Another interpretation of the number $a(G)$ gives the theorem of Tutte-Berge (cf. [7]). Since each matching is of even order, therefore the order n and the deficiency $a(G)$ of a graph G are of the same parity: $a(G) \equiv n (\mathrm{mod}\ 2)$.

In 1973 Mader [7] listed all graphs G each of which has a positive deficiency $a(G) > 0$ which decreases, when added any new edge if G is not complete. Hence one can easily obtain the characterization of graphs G of order n with the given deficiency $a(G) > 0$ which are maximal with respect to the relation: "to be a factor of" which coincides with the inclusion relation restricted in such a way that it holds true only between graphs with identical vertex sets.

Mader's result confirms the conjecture of Kotzig [6] formulated in 1969 on the structure of maximal graphs G without 1-factors (i.e., with $1 \leqslant a(G) \leqslant 2$). This structure was independently described by Homenko and Vyvrot [4] in 1971, and their result was improved by the present author [10] in 1973. The result deduced from that of Mader is more general and can be formulated as follows.

T h e o r e m. Let G be a maximal graph of order n with a given deficiency $a(G) > 0$. Then one has either

$$G = K_n \quad \text{if} \quad n \text{ is odd and } a(G) = 1$$

or there are integers h and k such that

(1.1) $$a(G) = k + 2 \geqslant 2$$

and

(1.2) $k \equiv n (\mathrm{mod}\ 2)$, $0 \leqslant k \leqslant n-2$, and $0 \leqslant h \leqslant (n-k-2)/2$, and there are $h+k+3$ mutually disjoint complete graphs $K_h^{(0)}$ and $K_{n_i}^{(i)}$, $i=1,2,\ldots,h+k+2$, where

$$(1.3)\qquad h+\sum_{i=1}^{h+k+2} n_i = n,\quad n_1 \geqslant n_2 \geqslant \ldots \geqslant n_{h+k+2} \geqslant 1,$$

and each n_i is an odd integer

such that G is of the form

$$(1.4)\qquad K_h^{(0)} * \sum_{i=1}^{h+k+2} K_{n_i}^{(i)}\ .$$

Conversely, each graph of the form (1.4) with integer parameters h, k, n_i, n satisfying (1.3) and (1.2) is one of the maximal graphs G on n vertices and with deficiency (1.1).

We omit the proof.

Observe that if G is of the form (1.4) and if $x \in V\left(K_h^{(0)}\right)$ (for $h>0$), then $d(x,G) = n-1$ (the maximal possible degree of vertices in G), while if $y \in V\left(K_{n_i}^{(i)}\right)$, then $d(y,G) = h + n_i - 1 < n - 1$. Moreover, the structure of the graph (1.4), i.e., its isomorphism type, is completely determined by the sequence of integers

$$h,\ n_1,\ n_2,\ \ldots,\ n_{h+k+2}$$

which satisfy (1.3). Observe that the sequence (n_i) is a partition of the positive integer $n-h$ into $h+k+2$ odd parts.

2. On partitions of integers

Partitions were considered already in the 17th century. Many interesting results on partitions were published by L. Euler [1] in a chapter "De partitione numerorum" and also in some subsequent papers. He introduced (factually formal) power series with one or two indeterminates as generating functions for unrestricted and for various restricted partitions of natural numbers. He proved certain identities involving those power series, found number-theoretical meanings of these identities, and derived recursion formulae for counting various partitions.

By a partition of the natural number s we mean a decreasing sequence $(m_1, m_2, \dots, m_r)$ of positive integers $m_1 \geqslant m_2 \geqslant \dots \geqslant m_r > 0$ (called parts or summands) whose sum is s. Denote by $p'(s,r)$ the number of partitions of s into r odd parts. Recall that the number of unrestricted partitions of s is usually denoted by $p(s)$, while $p_r(s)$ stands for the number of partitions of s with parts not exceeding r.

For the general theory of partitions the reader is referred to [2,3,8,9], an account on formal power series with one indeterminate (and possibly with additional parameter) can be found in [8] and [5]

Now we are going to outline the theory of formal power series with two indeterminates. Let $\mathbb{N} = \{0,1,\dots\}$ and $\mathbb{Z}$ denote the set as well as the ring of integers. Given an integral domain D (a commutative ring D with unity 1 and with no zero divisors, e.g., $D = \mathbb{Z}$) an integral domain $D[[t,z]]$ consists of doubly infinite arrays

$$u = (u_{kl}) \in D^{\mathbb{N} \times \mathbb{N}}$$

which are written as

$$u = u_{00} + u_{10}t + u_{01}z + u_{20}t^2 + u_{11}tz + u_{02}z^2 + \dots$$

and are called formal double power series over D with indeterminates t and z. The addition $(u,v) \to u+v$ and multiplication $(u,v) \to uv$ in $D[[t,z]]$ are defined in an usual way

$$(u+v)_{kl} = u_{kl} + v_{kl}\,, \quad (uv)_{kl} = \sum_{i=0}^{k} \sum_{j=0}^{l} u_{ij} v_{k-i,l-j}.$$

The multiplicative unit element in $D[[t,z]]$, denoted simply by 1, is series (u_{kl}) in which the leading coefficient

$u_{00} = 1$ (unity in D) and all the remaining ones are equal to 0 (zero of D). Note that the integral domain of (formal) polynomials in t,z over D is usually denoted by $D[t,z]$ and is a subring of $D[[t,z]]$. The integral domains of formal power series (and polynomials) over D with one or more than two indeterminates can be similarly defined.

Now the meanings of symbols $Z[[t]]$, $Z[t]$, $(Z[[t]])[[z]]$ etc. are clear. One can prove that the rings

$$(Z[[z]])[[t]],\quad Z[[t,z]],\ \text{and}\ (Z[[t]])[[z]]$$

are pairwise isomorphic. Given a doubly infinite array u, $u \in Z[[t,z]]$, let

$$u_{k.} = (u_{k0}, u_{k1}, \ldots) \quad \text{and} \quad u_{.l} = (u_{0l}, u_{1l}, \ldots)$$

be its k-th row and l-th column, respectively; and let $\overset{1}{u}$, $\overset{2}{u}$ be the series of rows of u and that of columns of u, respectively, i.e.,

$$\overset{1}{u} = (u_{0.}, u_{1.}, \ldots) \in (Z[[z]])[[t]]$$

$$\overset{2}{u} = (u_{.0}, u_{.1}, \ldots) \in (Z[[t]])[[z]] .$$

The correspondences

$$\overset{1}{u} \longleftrightarrow u \longleftrightarrow \overset{2}{u}$$

establish, as one can easily show, the isomorphisms in question. Thus u can be even identified either with $\overset{1}{u}$ or with $\overset{2}{u}$, and conversely.

One can show that a series u (with any number of indeterminates and with leading coefficient 1) is invertible in the corresponding ring of series, i.e., there is an inverse series u^{-1} (with leading coefficient equal to 1 too). The values of the function $v = u^{-1}$ (that is, the coefficients of the series u^{-1}) can be one after the other computed from the equality $vu = 1$.

An infinite sum $\sum_{i=0}^{\infty} u^{(i)}$ of formal power series $u^{(i)}$ is admitted if for any argument there is only a finite number of series $u^{(i)}$ each of which has a non-zero value at that argument. Similarly, an infinite product $\prod_{j=0}^{\infty} u^{(j)}$ of formal power series $u^{(j)}$ represents a power series if almost all factors $u^{(j)}$ have the value 1 at the argument $(0,0,\dots,0)$ and only a finite number of factors $u^{(j)}$ have non-zero values at any fixed non-zero argument (e.g., at any fixed $(k,l) \neq (0,0)$ in the case of series with two indeterminates).

The rule of substitution (cf. [8], p.212) can be extended over series with several indeterminates. Namely, the replacement of an indeterminate by any polynomial with constant term 0 is admissible as a transformation of a series into a series. Also the substitution of a constant for an indeterminate in a power series is admitted provided that it yields another power series with remaining indeterminates. So the Euler's method consisting in using the rule of substitution and deriving functional equations to prove certain identities (cf. the method (i) of § 19,5 in [3]) is justified as admissible also in the theory of formal power series. This method is widely used in [3] and [9], and can provide considerable simplifications in Rademacher's derivation of certain identities in §§ 96 and 97 of [8]. We shall use this method to prove another identity which is closely related to them. Put

$$G(t, z) = \prod_{i=0}^{\infty} \frac{1}{1-zt^{2i+1}} . \tag{2.1}$$

Clearly this infinite product represents a formal double power series which is the generating functions for the number $p'(s, r)$ of partitions of s into r odd parts, that is,

$$G(t,z) = 1 + \sum_{s=1}^{\infty} \sum_{\substack{1 \leq r \leq s \\ r+s \text{ even}}} p'(s,r) t^s z^r = 1 + \sum_{r=1}^{\infty} \sum_{\substack{s \geq r \\ r+s \text{ even}}} p'(s,r) t^s z^r . \tag{2.2}$$

Definition (2.1) implies that the ranges of s and r in (2.2) are correctly determined. Namely, it is easy to see that $p'(s,r) \neq 0$ iff it is true that

$$(2.3) \qquad s+r \text{ is even and } 1 \leqslant r \leqslant s \text{ or } s=0=r.$$

Moreover,

$$p'(s,s) = 1 \quad \text{for all integers} \quad s \geqslant 0$$

and

$$p'(s,1) = 1 \quad \text{for all odd} \quad s \geqslant 1.$$

Thus we can assume that

(2.4) $p'(0,0)=1$, and $p'(s,r)=0$ iff integers s,r do not satisfy (2.3).

The formula (2.1) also implies that $G(t,z)$ satisfies the following functional equation

$$(2.5) \qquad (1-zt)G(t,z) = G(t,zt^2).$$

The series $G(t,z)$ can be also represented as a series in z with coefficients, $D_r(t)$ say, from $\mathbb{Z}[[t]]$

$$(2.6) \qquad G(t,z) = \sum_{r=0}^{\infty} D_r(t)z^r \quad \text{with} \quad D_o(t) = 1.$$

Substitution of this series to the identity (2.5) gives the following identity

$$\sum_r \left(z^r D_r(t) - z^{r+1}\, t\, D_r(t)\right) = \sum_r z^r t^{2r} D_r(t).$$

Hence, equating coefficients of z^r, one obtains the recurrence formula

$$(2.7) \quad (1-t^{2r})\, D_r(t) = t\, D_{r-1}(t) \quad (r=1,2,\ldots) \quad \text{with} \quad D_o(t)=1,$$

whence

$$D_r(t) = \frac{t^r}{(1-t^2)(1-t^4)\dots(1-t^{2r})} \tag{2.8}$$

and, by (2.6) and (2.2),

$$D_r(t) = \sum_{\substack{s \geq r \\ s+r \text{ even}}} p'(s,r)t^s \qquad \text{for} \quad r = 1,2,\dots \tag{2.9}$$

Thus we have proved the following above-mentioned identity

$$G(t,z) \equiv \prod_{i=0}^{\infty} \frac{1}{1 - zt^{2i+1}} = 1 + \sum_{r=1}^{\infty} z^r \frac{t^r}{(1-t^2)(1-t^4)\dots(1-t^{2r})}$$

from which, putting $z = 1$ (what is admissible),we can obtain another identity.

The above results enable us to give the following number--theoretical interpretations of $p'(s,r)$. Namely, $p'(s,r)$ is the number of partitions:

(i) of s into r odd parts (with repetitions permitted);

(i′) of s into parts the maximal of which equals r and is the only part which appears an odd number of times in each of those partitions;

(ii) of $s-r$ into even parts not exceeding $2r$;

(ii′) of $s-r$ into at most $2r$ parts each of which appears an even number of times;

(iii) of $(s-r)/2$ into parts not exceeding r, i.e.,

$$p'(s,r) = p_r\left(\frac{s-r}{2}\right); \tag{2.10}$$

(iii′) of $(s-r)/2$ into at most r parts;

(iv) of $(s+r)/2$ with the maximal part r;

(iv′) of $(s+r)/2$ into r parts.

In fact, the interpretation (i) follows from (2.2) and (2.1), while (ii) follows from (2.9) and (2.8); (ii) in turn implies (iii) and (iv). Furthermore, each of interpretations marked by an undashed symbol, that is, (i), (ii), (iii), and (iv), is equivalent to that marked by the same symbol with a dash, for both of them concern so-called conjugate partitions. Recall that two conjugate partitions of a number n can be represented by one Ferrers-Sylvester diagram which is an array of n dots with rows representing parts in one of the two partitions and with columns corresponding to parts in the remaining one.

Regarding the formula (2.10), note that

$$F_r(t) := \frac{1}{(1-t)(1-t^2)\dots(1-t^r)} = \sum_{s=0}^{\infty} p_r(s)t^s$$

with $p_r(0) := 1$, $r = 1,2,\dots$, i.e., $F_r(t)$ is the generating series for $p_r(s)$. Hence, for $r=1$,

(2.11) $$p_1(s) = 1 \quad \text{for} \quad s = 0,1,\dots$$

For $r \geqslant 2$, $F_r(t)$ can be decomposed into partial fractions, each of which can be expanded into power series with complex coefficients, in general. Hence $p_r(s)$ can be obtained. Thus Rademacher [8] found that, for $s \geqslant 0$,

(2.12) $$p_2(s) = [s/2] + 1,$$

(2.13) $$p_3(s) = [s(s+6)/12] + 1,$$

where brackets denote integer parts.

For increasing r such formulae can also be found but they get more and more involved. For instance, we have

$$F_4(t) = \frac{1/24}{(1-x)^4} + \frac{1/8}{(1-x)^3} + \frac{25/144}{(1-x)^2} + \frac{1}{72}\,\frac{9x^2+17x+25}{1-x^3} + \frac{1}{16}\,\frac{1+x^2}{(1-x^2)^2} + \frac{1/8}{1+x} + \frac{1/8}{1+x^2} =$$

$$= \sum_{s=0}^{\infty} x^s\left(\frac{1}{24}\binom{s+3}{3} + \frac{1}{8}\binom{s+2}{2} + \frac{25}{144}(s+1)\right) + \frac{1}{72}\sum_{k=0}^{\infty}(9x^{3k+2}+17x^{3k+1}+25x^{3k}) +$$

$$+\frac{1}{16}\sum_{k=0}^{\infty}(2k+1)x^{2k}+\frac{1}{8}\sum_{k=0}^{\infty}\left(2x^{4k}-x^{4k+1}-x^{4k+3}\right).$$

Now one can obtain $p_4(s)$ as the coefficient of x^s in the above expansion of $F_4(t)$:

$$(2.14)\qquad p_4(s)=\left[s\left(s^2+15s+63+9\,\frac{1+(-1)^s}{2}\right)/144\right]+1.$$

In fifties of 19th century J.J.Sylvester developed a special method in order to compute the so-called denumerants, i.e., the numbers of partitions of a given integer into specified parts, e.g., he obtained (cf. [11]) that $p_3(s)$ (in his notation $\frac{s;}{1,2,3,}$) is the nearest integer to $(s+3)^2/12$, what is compatible with (2.13).

Now we shall represent the simplest method of evaluation, used by Euler already. It consists in deriving recurrence formulae. So substituting the series (2.9) into (2.7) one obtains

$$\sum_{\substack{s=r\\ s+r\ \text{even}}}\left(p'(s,r)t^s-p'(s,r)t^{s+2r}-p'(s,r-1)t^{s+1}\right)=0.$$

Hence, taking the coefficients of t^s, we have the following formula of recursion

$$(2.15)\qquad p'(s,r)=p'(s-2r,\ r)+p'(s-1,\ r-1)$$

with initial condition (2.4). The same result can be obtained by putting (2.2) into (2.5).

3. Counting graphs

We are interested in calculating

$$(3.1)\qquad \varphi_k(n,h):=p'(n-h,\ h+k+2)$$

which, according to Section 1, for parameters n, h, and k satisfying (1.2), is the number of maximal graphs with deficiency k+2 and with n vertices of which exactly h are of degree $n-1$. Therefore we change variables in (2.15) putting

$$\begin{cases} s = n-h \\ r = h+k+2, \end{cases}$$

where $n, h, k \in \mathbb{Z}$. Thus we obtain

$$p'(n-h,\ h+k+2) = p'(n-h-2(h+k+2),\ h+k+2) + p'(n-h-1,\ h+k+1).$$

Hence, by (2.15) and (2.4),

(3.2) $\varphi_k(n,h) = \varphi_k\big(n-2(h+k+2),h\big) + \varphi_k(n-2,h-1)$ for $k=\text{const.}\in \mathbb{Z}$

with initial condition

$$(3.3) \qquad \varphi_k(n,h) = 0$$

if and only if $h \notin \mathbb{Z}$ or $n+k$ is odd or neither $n \geqslant 2h+k+2 > h$ nor $n = h = -k-2$ holds true. Furthermore, $\varphi_k(-k-2,\ -k-2) = 1$.

The problem (3.2), (3.3) is clearly equivalent to the following one:

$\varphi_k(n,h) = \varphi_k\big(n-2(h+k+2),h\big) + \varphi_{k-1}(n-1,h)$ for $h=\text{const.} \in \mathbb{Z}$
with initial condition

$\varphi_k(n,h) = 0$ if $n+k$ is odd or neither $n \geqslant 2h+k+2 > h$ nor $n = h = -k-2$, and

$\varphi_k(h,\ h) = 1$ with $k = -h-2$.

Put

$$(3.4) \qquad f_k(n) = \sum_{h=0}^{(n-k-2)/2} \varphi_k(n,h).$$

Then, according to Section 1 and the formula (3.1), $f_k(n)$ is the number of isomorphism types of maximal graphs on n vertices and with the deficiency $k+2$, provided that $0 \leqslant k \leqslant n-2$. It is easy to see that, by the recursion formula (3.2),

$$(3.5) \qquad f_k(n) = \varphi_k\left(2n+k+2,\ \tfrac{n-k-2}{2}\right) - \varphi_k(n+2k+2,-1).$$

Moreover, since $\varphi_k(n,h) \geq 0$, therefore, by (3.4) and (3.3), $f_k(n)$ with $k \in \mathbb{N}$ is non-zero only if $n+k$ is even and $0 \leq k \leq n-2$. In general, $f_k(n) \neq 0 \Leftrightarrow k \in \{-n-2, -n, -n+2, \ldots, n-2\}$ and $n \in \mathbb{N} = \{0,1,\ldots\}$.

In order to simplify the recurrence equation (3.2) we put

$$n+k+2 = \xi, \quad h+k+2 = \eta \text{ (with } \xi \in \mathbb{Z})$$

and

$$\psi(\xi,\eta) := \varphi_k(\xi-k-2, \eta-k-2). \tag{3.6}$$

Then the problem (3.2), (3.3) reduces to the recurrence equation

$$\psi(\xi,\eta) = \psi(\xi-2\eta, \eta) + \psi(\xi-2, \eta-1) \quad (\text{where } \xi \in \mathbb{Z}) \tag{3.7}$$

with initial condition

$$\psi(0,0) = 1, \tag{3.8}$$

and, furthermore, $\psi(\xi,\eta) = 0$ iff $\eta \notin \mathbb{Z}$ or ξ is odd or neither $\xi \geq 2\eta > 0$ nor $\xi = \eta = 0$.

Hence it follows that we actually may consider ψ as the function of two variables only. So, by (3.6),

$$\varphi_k(n, h) = \psi(n+k+2, h+k+2), \tag{3.9}$$

and, by (3.4),

$$f_k(n) = \sum_{h=0}^{(n-k-2)/2} \psi(n+k+2, h+k+2) = \sum_{\eta=k+2}^{(n+k+2)/2} \psi(n+k+2, \eta). \tag{3.10}$$

Moreover, by (3.5) and (3.9), or by (3.10) and (3.7), we get

$$f_k(n) = \psi(2n+2k+4, (n+k+2)/2) - \psi(n+3k+4, k+1). \tag{3.11}$$

Furthermore, by (3.6), (3.1), and (2.10), we have

$$\psi(\xi,\eta) = \varphi_k(\xi-k-2, \eta-k-2) = p'(\xi-\eta, \eta) = p_\eta((\xi-2\eta)/2).$$

Hence, by (2.11), (2.12), (2.13), and (3.8),

$$(3.12)\quad \begin{cases} \psi(\xi,\ 1) = 1 & \text{for } \xi = 2,4,6,\dots, \\ \psi(\xi,\ 2) = [\xi/4] & \text{for even } \xi \geqslant 0, \\ \psi(\xi,\ 3) = [(\xi^2+12)/48] & \text{for even } \xi \geqslant -4. \end{cases}$$

Using (2.14) one can get the explicit formula for $\psi(\xi,4)$ too.

Consider (3.1) and (3.3) as the definition of $\varphi_k(n,h)$ for any $k,\ n \in \mathbb{Z}$ and $h \in \mathbb{R}$. Then, in particular, by (3.6),

$$\psi(n,\ h) = \varphi_{-2}(n,\ h).$$

Moreover, the formulae (3.9), (3.11), and (3.12) imply that

$$\varphi_k(n,h) = \varphi_j(n+k-j,\ h+k-j),$$

$$f_k(n) = \varphi_j(2n+2k-j+2,\ (n+k-2j-2)/2) - \varphi_j(n+3k-j+2,\ k-j-1),$$

$$\varphi_k(n,\ -k-1) = 1 \qquad \text{for } n=-k,\ -k+2,\ -k+4,\dots,$$

$$\varphi_k(n,\ -k) = [(n+k+2)/4] \qquad \text{for } n=-k-2,\ -k,\ -k+2,\dots,$$

$$\varphi_k(n,\ -k+1) = \left[\left((n+k+2)^2+12\right)/48\right] \text{for } n=-k-6,\ -k-4,\ -k-2,\dots$$

Observe that, by (3.11), (3.12), and (3.3),

$$f_0(n) = \psi(2n+4,\ (n+2)/2) - 1 \qquad \text{for even } n \geqslant -2,$$

$$f_1(n) = \psi(2n+6,\ (n+3)/2) - [(n+7)/4] \qquad \text{for odd } n \geqslant -7,$$

$$f_2(n) = \psi(2n+8,(n+4)/2) - [(n^2+20n+112)/48] \text{ for even } n \geqslant -14.$$

Moreover, it follows that

$$f_1(n) = f_0(n+1) - [(n+3)/4] \qquad \text{for odd } n \geqslant -3,$$

$$f_2(n) = f_1(n+1) + [(n+8)/4] - [(n^2+20n+112)/48] \quad \text{for even } n \geqslant -8,$$

$$= f_0(n+2) - [(n^2+20n+64)/48] \qquad \text{for even } n \geqslant -4.$$

The formulae (3.9) and (3.10) express both $\varphi_k(n,h)$ and $f_k(n)$ by means of the values $\psi(\xi,\eta)$ of the function ψ. An effective tabulation of $\psi(\xi,\eta)$ can be performed using recurrence formula (3.7) together with (3.8). The non-zero values $\psi(\xi,\eta)$ for $\xi \leqslant 24$ (and consequently $\eta \leqslant 12$) are given in the coordinate system $O\,\xi\,\eta$ in Table 1. In this table the values

$$\psi(4n,\ n) = \sum_{\eta} \psi(2n,\eta) = f_0(2n-2) + 1 \quad (\text{with } n \in \mathbb{N})$$

are put into small squares. Furthermore, above the line $\eta \doteq \xi$ there are numbers 0,1,...,10 which are not the values of ψ. Any of them, k say, corresponds to a point, say O'_k, which is the nearest of the distinguished points on the line $\eta = \xi$. Then, when we translate the origine O of the coordinate system $O\,\xi\,\eta$ to O'_k, we obtain a new coordinate system, $O'_k\,\xi'\,\eta'$ say, in which, according to (3.6), the values given in the Table 1 can be read off as the values $\varphi_k(\xi',\eta')$ of φ_k.

Table 1

η \ ξ	0	2	4	6	8	10	12	14	16	18	20	22	24
12							10						1
11						9						1	1
10						8					1	1	2
9					7					1	1	2	3
8					6				1	1	2	3	5
7				5				1	1	2	3	5	7
6				4			1	1	2	3	5	7	[11]
5			3			1	1	2	3	5	[7]	10	13
4			2		1	1	2	3	[5]	6	9	11	15
3		1		1	1	2	[3]	4	5	7	8	10	12
2		0	1	1	[2]	2	3	3	4	4	5	5	6
1		1	[1]	1	1	1	1	1	1	1	1	1	1
0	[1]												

Table 2 gives all non-zero values of $f_k(n)$ for $2 \leqslant k+2 \leqslant n \leqslant 25$ and $k \leqslant 10$. The values lying on the left of the sloped line can be obtained from Table 1 using the formula (3.10). Table 2 gives the numbers of maximal graphs of order $n \leqslant 25$ and with deficiency $k+2$, where $0 \leqslant k \leqslant 10$.

Table 2

k \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	1		2		4		6		10		14		21		29		41		55		76		100	
1		1		2		4		7		11		17		25		36		50		70		94		127
2			1		2		4		7		12		18		28		40		58		80		111	
3				1		2		4		7		12		19		29		43		62		88		122
4					1		2		4		7		12		19		30		44		65		92	
5						1		2		4		7		12		19		30		45		66		95
6							1		2		4		7		12		19		30		45		67	
7								1		2		4		7		12		19		30		45		67
8									1		2		4		7		12		19		30		45	
9										1		2		4		7		12		19		30		45
10											1		2		4		7		12		19		30	

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