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Mirosław Pustelnik

SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A SCALAR PRODUCT ON A GENERALIZED MODULE OF VECTOR FIELDS

Introduction

The present paper contains results concerning the existence of a scalar product on the differentiation module of some associative algebra over an associative commutative ring with unity.

The first part of this work is devoted to the investigation of algebraic counterparts for the notions of contraction and localization of functions from a differential structure. We also investigate the problem of inducing vector fields on a differential subspace of a given differential space and give a description of orientation of a ring together with an order relation induced by this orientation.

In the second part of this work we consider a smooth decomposition of unity in the ring and a scalar product defined on the differentiation module Diff C of some associative algebra C over an associative and commutative ring with unity. Sufficient conditions are formulated in order that there exist a scalar product and a symmetric covariant derivative on the module Diff C.

The obtained results and constructions are interpreted in a differential space as well as in a differentiable manifold.

I. AN ALGEBRAIC THEORY OF INDUCED VECTOR FIELDS AND INDUCED ORDERINGS OF THE RING

1. The operation of contraction

Let R be an associative and commutative ring with unity, and let C be any associative R-algebra. Assume that the ring R, treated as an R-algebra, is a subalgebra of the algebra C. Let A be any set of homomorphisms $p: C \to R$ such that $p(r) = r$ for $r \in R$. For any $\alpha \in C$ we define the function $A^{o}(\alpha)$ on the set A by the formula

$$A^{o}(\alpha)(p) = p(\alpha) \qquad \text{for all } p \in A. \tag{1}$$

Then for all $\alpha, \beta \in C$ we have the formulas

$$\begin{cases} A^{o}(\alpha+\beta) = A^{o}(\alpha) + A^{o}(\beta) \\ A^{o}(\alpha\cdot\beta) = A^{o}(\alpha)\cdot A^{o}(\beta) \\ A^{o}(r) = r_A, \end{cases} \tag{2}$$

where r_A denotes the constant function with domain A, equal to r for all $p \in A$.

Let $F(A,R)$ denotes the R-algebra of all functions $\gamma: A \to R$, where for all $p \in A$

$$\begin{cases} (\gamma+\gamma')(p) = \gamma(p) + \gamma'(p) \\ (\gamma\cdot\gamma')(p) = \gamma(p)\cdot\gamma'(p) \\ (r\cdot\gamma)(p) = r\cdot\gamma(p). \end{cases}$$

Hence we have

$$A^{o}: C \to F(A,R). \tag{3}$$

Let C_A be a subalgebra of the R-algebra $F(A,R)$. We shall consider the homomorphism

(4) $$\hat{A}: C \to C_A$$

such that $\hat{A}(\alpha) = A(\alpha)$ for all $\alpha \in C$.

2. The case of a real associative algebra

Let R be the field of real numbers, and let C be a differential structure over a set M. C is clearly an associative R-algebra. If we identify real numbers $r \in R$ with constant functions on M: $r(x) = r$ for all $x \in M$, then R is a subalgebra of the algebra C. With every point $x \in M$ we associate the homomorphism $h(x): C \to R$ defined by the formula

(5) $$h(x)(\alpha) = \alpha(x) \quad \text{for all } \alpha \in C.$$

Then

$$h(x)(r) = r(x) = r \quad \text{for } r \in R.$$

Let $E = h[M] = \{h(x);\ x \in M\} \subset \mathrm{Hom}\,(C,R)$ and let τ_C denote the topology on M induced by the set C of real functions. Then we have the following theorem.

T h e o r e m 1. The mapping $h: M \to E$ is one-to-one iff (M, τ_C) is a Hausdorff space.

P r o o f. Let (M,τ_C) be a Hausdorff space and let $h(x_1) = h(x_2)$, i.e. $h(x_1)(\alpha) = h(x_2)(\alpha)$ for all $\alpha \in C$. This implies $\alpha(x_1) = \alpha(x_2)$ for every α, and by assumption it follows that $x_1 = x_2$ (see [3] p.69). Now suppose that (M,τ_C) is not a Hausdorff space. Hence there are points $x_1, x_2 \in M$, $x_1 = x_2$ such that for every function $\alpha \in C$ we have $\alpha(x_1) = \alpha(x_2)$. This implies $h(x_1)(\alpha) = h(x_2)(\alpha)$, which means that the mapping h is not one-to-one. In the sequel we shall assume that (M,τ_C) is a paracompact space, which guarantees that h is one-to-one.

Let $x \in U \in \tau_C$. Then $h[U] = A \subset E$. For any $\alpha \in C$ we obtain from (1), the following equality

$$A^o(\alpha)\big(h(x)\big) = h(x)(\alpha) = \alpha(x),$$

i.e. $A^{o}(\alpha) \circ h | U = \alpha | U$, or in the equivalent form

(6) $$A^{o}(\alpha) = \alpha | U \circ h^{-1} | h[U] = \alpha | h^{-1}[A] \circ h^{-1} | A = \alpha \circ h^{-1} | A.$$

Now the formulas (2) can be easily verified. Thus we have for example

$$A^{o}(\alpha+\beta) = (\alpha+\beta) \circ h^{-1} | A = \alpha \circ h^{-1} | A + \beta \circ h^{-1} | A = A^{o}(\alpha) + A^{o}(\beta)$$

and

$$A^{o}(r) = r \circ h^{-1} | A = r_A.$$

Let U be a subset of M. In agreement with [3], by $C_{|U}$ we denote the set of all local C-functions defined on U. For any $A \subset E$ let

(7) $$C_A = \left\{ \beta \circ h^{-1} | A ;\ \beta \in C_{h^{-1}[A]} \right\}.$$

Thus if $\alpha \in C$, then $\alpha | h^{-1}[A] \in C | h^{-1}[A] \subset C_{h^{-1}[A]}$ and moreover we have

$$\alpha | h^{-1}[A] \circ h^{-1} | A = \alpha \circ h^{-1} | A \in C_A.$$

From formulas (3) and (4), it follows that we can define a homomorphism $\hat{A}: C \to C_A$ by the formula

(8) $$\hat{A}(\alpha) = \alpha \circ h^{-1} | A.$$

Then in particular for $r \in R$ we have

(9) $$A(r) = r \circ h^{-1} | A = r_A.$$

3. An algebraic theory of inducing vector fields

By Diff C we shall denote the C-module of all differentiations of the R-algebra C; i.e. the set of all R-linear maps

$X: C \to C$ satisfying the condition $X(\alpha \cdot \beta) = X(\alpha)\cdot\beta + \alpha\cdot X(\beta)$. The operations in Diff C are defined as follows

$$(10)\quad \begin{cases} (X+Y)(\alpha) = X(\alpha) + Y(\alpha) & \text{for } X,Y \in \text{Diff } C,\ \alpha \in C \\ (\varphi\cdot X)(\alpha) = \varphi\cdot X(\alpha), & \text{for } X \in \text{Diff } C,\ \alpha,\varphi \in C. \end{cases}$$

Let A be any set contained in Hom (C,R) and let for every $X \in$ Diff C there exist exactly one element $A'(X) \in$ Diff C_A such that the following diagram is commutative

$$(11)\quad \begin{array}{ccc} C & \xrightarrow{X} & C \\ \hat{A}\downarrow & & \downarrow\hat{A} \\ C_A & \xrightarrow{A'(X)} & C_A \end{array}$$

Let (C) denote the family of all sets $A \subset \text{Hom}(C,R)$ satisfying the above condition and such that $p(r) = r$ for $r \in R$ and $p \in A$. Thus for every $A \in (C), \alpha \in C$ we have

$$(12)\quad A'(X)(\hat{A}(\alpha)) = \hat{A}(X(\alpha)).$$

This implies

$$A'(X+Y)(\hat{A}(\alpha)) = \hat{A}(X(\alpha) + Y(\alpha)) = \hat{A}(X(\alpha)) + \hat{A}(Y(\alpha)) =$$

$$= A'(X)(\hat{A}(\alpha)) + A'(Y)(\hat{A}(\alpha)) = (A'(X) + A'(Y))(\hat{A}(\alpha)) .$$

Now it follows from (11) that

$$(13)\quad A'(X+Y) = A'(X) + A'(Y) .$$

In a similar way we verify that

$$(14)\quad A'(\beta\cdot X) = A^{o}(\beta)\cdot A'(X),$$

hence in particular, for $X,Y \in$ Diff C, we have

$$(15)\quad A'(r\cdot X) = r\cdot A'(X)$$

Hence we have the following theorem.

T h e o r e m 2. For every $A \in (C)$ the mapping $A': \mathrm{Diff}\ C \to \mathrm{Diff}\ C_A$ is R-linear.

4. The case of vector fields over a differential space

First we shall prove the following lemma.

L e m m a 1. If $Z \in \mathrm{Diff}\ C_A$, $p \in h[V] \subset A$, where $V \in \tau_C$, and if for $\beta_o \in C_{h^{-1}[A]}$

$$\beta_o \mid V = 0, \tag{16}$$

then

$$Z(\beta_o \circ h^{-1}|A)(p) = 0.$$

P r o o f. Let us put

$$Y(\beta) = Z(\beta \circ h^{-1}|A) \circ h|h^{-1}[A] \quad \text{for } \beta \in C_{h^{-1}[A]}.$$

Then for any $\beta, \gamma \in C_{h^{-1}[A]}$ we have

$$Y(\beta+\gamma) = Z\big((\beta+\gamma)\circ h^{-1}|A \circ h|h^{-1}[A] = Z(\beta\circ h^{-1}|A + \gamma\circ h^{-1}|A)\circ h|h^{-1}[A] =$$

$$= Z(\beta\circ h^{-1}|A)\circ h|h^{-1}[A] + Z(\gamma\circ h^{-1}|A)\circ h|h^{-1}[A] = Y(\beta) + Y(\gamma).$$

and

$$Y(\beta\cdot\gamma) = Z\big((\beta\cdot\gamma)\circ h^{-1}|A \circ h|h^{-1}[A] =$$

$$= Z(\beta\circ h^{-1}|A \cdot(\gamma\circ h^{-1}|A) + (\beta\circ h^{-1}|A)\cdot Z(\gamma\circ h^{-1}|A) \circ h|h^{-1}[A] =$$

$$= Y(\beta)\cdot\gamma + \beta\cdot Y(\gamma).$$

and

$$Y(r\cdot\beta)=Z\big((r\cdot\beta)\circ h^{-1}|A)\circ h|h^{-1}[A] = r\cdot Z(\beta\circ h^{-1}|A)\circ h|h^{-1}[A] =$$

$$= r\cdot Y(\beta).$$

This implies that $Y\in \mathrm{Diff}\, C_{h^{-1}[A]}$, and finally we have

$$0 = Y(\beta_0)\, h^{-1}(p)\big) = \big(Z(\beta_0\circ h^{-1}|A)\circ h|h^{-1}[A]\big)\big(h^{-1}(p)\big) =$$

$$= Z(\beta_0\circ h^{-1}|A)(p).$$

Let $(C) = \left\{A\subset E;\ h^{-1}[A]\in\tau_C\right\}$. Now we can prove the following theorem.

T h e o r e m 3. If $A\in(C)$, then for every $X\in\mathrm{Diff}\, C$ there exists exactly one $A'(X)\in\mathrm{Diff}\, C_A$ such that $A'(X)\big(\hat{A}(\alpha)\big) = \hat{A}\big(X(\alpha)\big)$ for $\alpha\in C$.

P r o o f. First we shall define the element $A'(X)$. Let $f\in C_A$, thus $f=\beta\circ h^{-1}|A$, where $\beta\in C_{h^{-1}[A]}$. For an arbitrary point $p\in A$ there exists a neighbourhood $V\in\tau_C$ of the point $h^{-1}(p)$ and a function $\gamma\in C$ such that

$$\gamma|V\cap h^{-1}[A] = \beta|V\cap h^{-1}[A].$$

Hence we can put

$$A'(X)(f)(q) = \big(X(\gamma)\circ h^{-1}|A\big)(q) \quad \text{for} \quad q\in h[V]\cap A. \tag{17}$$

In particular, if $\alpha\in C$, then $\alpha\,|h^{-1}[A]\in C_{h^{-1}[A]}$. Moreover,

$$\alpha|h^{-1}[A]\circ h^{-1}|A = \alpha\circ h^{-1}|A\in C_A.$$

As γ we can take the function α and $V = h^{-1}[A]$. Hence we have

$$\big(A'(X)\big(\hat{A}(\alpha)\big)\big)(p) = \big(X(\alpha)\circ h^{-1}|A)(p) \quad \text{for} \quad p\in A;$$

that is

$$A'(X)\big(\hat{A}(\alpha)\big) = \hat{A}\big(X(\alpha)\big). \tag{18}$$

Now let us take an arbitrary function $\alpha \in C_A$ and a point $p \in A$. Let $x = h^{-1}(p)$, $U = h^{-1}[A]$, then $x \in U$ and $\alpha \circ h|h^{-1}[A] = \beta \in C_U$ (as well as $\alpha = \beta \circ h^{-1}|A \in C_A$). There exists $V \in \tau_C$ such that $x \in V \subset U$, and there exists a function $\bar{\beta} \in C$ such that $\beta|V = \bar{\beta}|V$, which implies $\beta|V = (\bar{\beta}|U)|V$.

Let Z, Z' be any elements of Diff C_A such that the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{X} & C \\ \hat{A}\downarrow & & \downarrow\hat{A} \\ C_A & \xrightarrow{Z} & C_A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{X} & C \\ \hat{A}\downarrow & & \downarrow\hat{A}. \\ C_A & \xrightarrow{Z'} & C_A \end{array}$$

In agreement with Lemma 1 we put $\beta_o = \beta - \bar{\beta}|U$ und we obtain

$$0 = Z(\beta_o \circ h^{-1}|A)(p) = Z(\beta \bullet h^{-1}|a)(p) - Z(\bar{\beta}|U \circ h^{-1}|A)(p),$$

which implies

$$Z(\beta \circ h^{-1}|A)(p) = Z(\beta_o \circ h^{-1}|A)(p) = Z\big(\hat{A}(\beta)\big)(p) = \hat{A}\ X(\beta)\big)(p).$$

As a consequence we have

$$Z(\alpha)(p) = \hat{A}\big(X(\bar{\beta})\big)(p), \quad Z'(\alpha)(p) = \hat{A}\big(X(\bar{\beta})\big)(p).$$

Hence $Z(\alpha)(p) = Z'(\alpha)(p)$ for $p \in A$ - and $Z(\alpha) = Z'(\alpha)$ for $\alpha \in C_A$, thus finally $Z = Z'$.

5. Oriented rings

Let, as before, R be a commutative and associative ring with unity, under the operations + and $\cdot$. By orientation of the ring R we shall understand any endomorphism $\varepsilon : (R, \cdot) \to (R, \cdot)$ satisfying the following conditions

$$(19) \quad \begin{cases} \text{(a)} \quad \varepsilon(r+r') = \varepsilon(r) \text{ whenever } \varepsilon(r') = 0 \text{ or } \varepsilon(r') = \varepsilon(r) = 1, \\ \text{(b)} \quad \varepsilon(0) = 0, \ \varepsilon(-1) \neq 1. \end{cases}$$

The ring R together with an orientation will be called an oriented ring.

The choice of orientation ε allows us to introduce a "less than" relation $<$ between elements of R, induced by the orientation ε. Namely, we define $0<r$ iff $\varepsilon(r) = 1$, and next $r_1<r_2$ iff $0<r_2 - r_1$. If $r_1<r_2$ or $r_1 = r_2$ we write $r_1 \leqslant r_2$. We shall prove the following theorem.

T h e o r e m 4. If (R,ε) is an oriented ring, then the relation $<$ induced by the orientation ε is antireflexive, antisymmetric, transitive and the following conditions are satisfied

$$(20) \quad \begin{cases} \text{(i) if } 0<r_1 \text{ and } 0<r_2, \text{ then } 0<r_1\cdot r_2; \\ \text{(ii) if } 0\leqslant r_1 \text{ and } 0<r_2, \text{ then } 0<r_1+r_2. \\ \text{(iii) if } r_1<r_2, \text{ then } r_1+r<r_2+r \text{ for any } r\in R. \\ \text{(iv) if } r_1<r_2 \text{ and } 0<r, \text{ then } r_1\cdot r<r_2\cdot r. \end{cases}$$

P r o o f. From (b) it follows that $0 = \varepsilon(0) = \varepsilon(r-r)$. Hence $\varepsilon(r-r) \neq 1$, and it is not true that $r<r$. If $r_1<r_2$, then $\varepsilon(r_1-r_2) = \varepsilon\big((-1)(r_2-r_1)\big) = \varepsilon(-1)\varepsilon(r_2-r_1) = \varepsilon(-1)\cdot 1 = \varepsilon(-1) \neq 1$, thus it is not true that $r_2<r_1$. Assume in addition that $r_2<r_3$. Then by (a) we have $\varepsilon(r_3 - r_1) = \varepsilon\big((r_3-r_2) + (r_2-r_1)\big) = \varepsilon(r_3-r_2) = 1$, i.e. $r_1<r_3$. Thus the relation is antireflexive, antisymmetric and transitive. Now assume that $0<r_1$ and $0<r_2$. Then $\varepsilon(r_1\cdot r_2) = \varepsilon(r_1)\varepsilon(r_2) = 1\cdot 1 = 1$. Hence condition (i) holds. If $0\leqslant r_1$ and $0<r_2$, than $\varepsilon(r_1) = 0$ or else $\varepsilon(r_1) = \varepsilon(r_2) = 1$. This implies $\varepsilon(r_1+r_2) = \varepsilon(r_2) = 1$, i.e. $0<r_1+r_2$. Condition (iii) follows from the identity $(r_2+r)-(r_1+r) = r_2-r_1$, and condition (iv) follows from the identity $r_2r - r_1r = (r_2-r_1)r$ and from the fact that ε is an endomorphism.

E x a m p l e. Let $(R,+,\cdot)$ be the ring of integers (or real numbers) and let

$$\varepsilon(x) = \operatorname{sgn} x \quad \text{for} \quad x\in R.$$

The function $\operatorname{sgn} x$ is clearly an endomorphism of the semigroup $(R,\cdot)$ and $\varepsilon[R] = \{-1,0,1\}$. Besides that we have

(a) if $\operatorname{sgn} x' = 0$, then $x' = 0$ and $\operatorname{sgn}(x+x') = \operatorname{sgn} x$; and similarly, if $\operatorname{sgn} x = \operatorname{sgn} x' = 1$, then $\operatorname{sgn}(x+x') = 1$.

(b) $\operatorname{sgn}(0) = 0$; $\operatorname{sgn}(-1) = -1 \neq 1$.

Hence the function sgn is an orientation of the ring R and it induces in R a "less-than" relation identical with the usual "less-than" relation between integers (real number).

With a given ring R and a set A one can associate the algebra R^A of all functions defined on A with values in R by defining in the usual way (pointwise) the operations of addition, multiplication and multiplication by the elements of the ring R.

If (R,ε) is an oriented ring, then we can introduce in R the "less-than" relation. This relation next allows us to introduce in R^A and analogous relation, called the "less--than" relation in R^A induced by ε. We shall denote this relation by the same symbol $<$ without misunderstanding. We define it as follows

(21) $f<g$ $(f\leqslant g)$ iff for every $x\in R$ we have $f(x)<g(x)$ $(f(x)\leqslant g(x))$.

If we denote by 0 the function 0_A (i.e. the function defined on A and taking everywhere the value 0 of the ring R), then we can derive from Theorem 4 the following corollary.

C o r o l l a r y. The "less-than" relation $<$ induced by ε in R^A, is antireflexive, antisymmetric and transitive, as well as it satisfies the conditions

(22)
- (i') if $0<f_1$ and $0<f_2$, then $0<f_1\cdot f_2$;
- (ii') if $0\leqslant f_1$ and $0<f_2$, then $0<f_1+f_2$;
- (iii') if $f_1<f_2$, then $f_1+f<f_2+f$ for any $f\in R^A$;
- (iv') if $f_1<f_2$ and $0<f_1$ then $f_1\cdot f<f_2\cdot f$;
- (v') if $f_1<f_2$ and $0<r\in R$, then $r\cdot f_1<r\cdot f_2$;
- (vi') if $0\leqslant f_i$ for $i\in\{1,\dots,k\}$ and $0<f_j(x)$ for some $j\in\{1,\dots,k\}$, then $0<f_1+\dots+f_k$.

P r o o f. If $0<f_1$ and $0<f_2$, then for any $x\in A$ we have $0<f(x)$ and $0<f_2(x)$, hence from condition (ii) of Theorem 4 it follows that $0<f_1(x)\cdot f_2(x)$; i.e. condition (i') holds. Similarly one can prove conditions (ii'), (iii') and (iv'). If for any $x\in A$ we have $f_1(x)<f_2(x)$ and $0<r\in R$, then $r\cdot f_1(x)<r\cdot f_2(x)$ by condition (iv) of Theorem 4. Hence condition (v') holds. Condition (vi') follows directly from condition (ii) of Theorem 4 and from condition (ii').

II. ALGEBRAIC CONDITIONS FOR THE EXISTENCE OF A SCALAR PRODUCT AND A COVARIANT DERIVATIVE

1. Smooth decomposition of unity in a commutative ring

Let E denote a fixed subset of the set Hom (C,R) (see I, § 1). Let $A\subset E$ be a set with the property that for every $p\in A$ there is exactly one homomorphism

$$p_A:\ C_A \longrightarrow R \tag{1}$$

such that

$$p_A\circ \hat{A} = p. \tag{2}$$

For any $\eta\in C_A$ and $\alpha\in C$, let

$$(A\alpha)(\eta)(p) = \begin{cases} p(\alpha)\cdot p_A(\eta) & \text{if } p\in A \\ 0 & \text{if } p\in E-A. \end{cases} \tag{3}$$

Hence we have obtained an element $(A\alpha)(\eta)$ belonging to $F(E,R)$.

We say that an element α in the R-algebra C is subject to the set A, if for any function $\eta\in C_A$ there exists exactly one element $\beta\in C$ such that $\check{\beta} = (A\alpha)(\eta)$ where

$$\check{\beta}(p) = p(\beta) \quad \text{for } p\in E. \tag{4}$$

If a function $\varphi : E \to R$ has the property that there exists exactly one element $\beta \in C$ such that $\check{\beta} = \varphi$, then this element will be denoted by $[\varphi]$.

Let $\check{C}(E)$ denote the set of all functions $\varphi \in F(E,R)$ satisfying the condition: there exists exactly one element $\beta \in C$ such that $\check{\beta} = \varphi$.

T h e o r e m 1.1. If the set $\check{C}(E)$ is closed under the operations of addition, multiplication, and multiplication by scalars in the R-algebra $F(E,R)$, then the set $C(E)$ of all elements of the form $[\varphi]$, where $\varphi \in \check{C}(E)$ is closed in the R-algebra C. Hence $C(E)$ is a subalgebra of the R-algebra C and the map

$$\varphi \longmapsto [\varphi] : \check{C}(E) \to C(E) \tag{5}$$

is an isomorphism.

P r o o f. The first part of the theorem is obvious. Assume that the set $\check{C}(E)$ is closed in the R-algebra $F(E,R)$ and let $\alpha, \beta \in C(E)$, $r \in R$. Then $\alpha = [\varphi], \beta = [\psi]$ where $\varphi, \psi \in C(E)$. We thus have $\check{\alpha} = \varphi, \check{\beta} = \psi$. This implies

$$(\alpha+\beta)\check{}(p) = p(\alpha+\beta) = p(\alpha) + p(\beta) = \check{\alpha}(p) + \check{\beta}(p) = \varphi(p) + \psi(p) =$$

$$= (\varphi+\psi)(p)$$

for $p \in E$. Hence $(\alpha+\beta)\check{} = \varphi + \psi$. By assumption, $\varphi + \psi \in \check{C}(E)$, that is

$$[\varphi + \psi] = [\varphi] + [\psi].$$

Similarly one verifies that the following equalities hold

$$[\varphi \cdot \psi] = [\varphi]\cdot[\psi], \quad [r \cdot \varphi] = r \cdot [\varphi].$$

Hence the map (5) is a homomorphism. If $\varphi \in C(E)$ and $[\varphi] = 0$, then $\check{0} = \varphi$. But $\check{0}(p) = p(0) = 0$ for $p \in E$, hence $\varphi = 0$ (i.e. it is the zero of the R-algebra $F(E,R)$).

Let (R,ε) be an oriented ring. Consider the family $\mathfrak{A}$ of the set E with the property

(*) for every $A \in \mathfrak{A}$ and for any $p \in A$ there exists exactly one homomorphism of the form (1) such that (2) holds.

An indexed family $(\varphi_s;\ s \in S)$ of elements of C will be called a smooth decomposition of unity subject to the family $\mathfrak{A}$, if there exists a function

$$s \longmapsto A_s \ :\ S \longrightarrow \mathfrak{A}$$

such that

(i) $$\bigcup_{s\in S}\left\{p;\ p \in A_s \wedge p(\varphi_s) \neq 0\right\} = E;$$

(ii) $\left\{s : p(\varphi_s) \neq 0\right\}$ is a finite set for $p \in E$.

(iii) for any indexed family $(\eta_s:\ s \in S) \in \mathbb{P}_{s\in S} C_{A_s}$ there exists exactly one element $\beta \in C$ such that

(6) $$\breve{\beta} = \sum_{s\in S} (A_s\varphi_s)(\eta_s)$$

(iv) $$0 \leqslant \breve{\varphi}_s \quad \text{for} \quad s \in S$$

(v) $$\sum_{s\in S} p(\varphi_s) = 1 \quad \text{for} \quad p \in E.$$

A function $s \longmapsto A_s$ satisfying conditions (i)-(iv) will be called a choice function for the given smooth decomposition of unity.

Observe that in view of (ii) the definitions of sums appearing in (iii) and (v) are well formulated. In fact, for any $p \in E$ the set of all $s \in S$ for which $(A_s\,\varphi_s)(\eta_s)(p) \neq 0$ is, according to (1), contained in $\left\{s;\ p(\varphi_s)\cdot p_{A_s}(\eta_s) \neq 0\right\} \subset \left\{s;\ p(\varphi_s) \neq 0\right\}$, and the latter is finite.

2. Smooth decomposition of unity in a differential space

Let (M,C) be a differential space. We shall assume that the topological space (M,τ_C) is paracompact and C-normal, i.e. for any disjoint closed sets F and H there exists $\alpha \in C$ such that $\alpha|F = 1_F$, $\alpha|H = 0_H$, $\alpha \geqslant 0$.

Let us take an arbitrary set $A \subset E$. To every point $x \in h^{-1}[A] \subset M$ we associate the homomorphism $h_A(x)\colon C_A \longrightarrow R$, defined by the formula

$$h_A(x)(\alpha) = \alpha\bigl(h(x)\bigr), \quad \text{for } \alpha \in C_A.$$

Next we denote $p_A = h_A(x)$. In this way to every point $p \in A$ there corresponds a homomorphism $p_A\colon C_A \longrightarrow R$ where $p_A(\alpha) = \alpha(p)$ for $p \in A$ and $\alpha \in C_A$. For every $\beta \in C$ we have

$$(p_A \circ \hat{A})(\beta) = p_A(\beta \circ h^{-1}|A) = \bigl((h^{-1}|A)(p)\bigr) = \beta\bigl(h^{-1}(p)\bigr) = p(\beta).$$

Hence

$$p_A \circ \hat{A} = p.$$

and equality (2), § 1, holds.

Next let $\alpha \in C$, and let $A \subset E$ be any set such that $h^{-1}[A] \in \tau_C$. For arbitrary $\eta \in C_A$ and $p \in A$ we have

$$p(\alpha)\cdot p_A(\eta) = \alpha\bigl(h^{-1}(p)\bigr)\cdot\eta(p) = \bigl((\alpha \circ h^{-1}|A)\cdot\eta\bigr)(p).$$

Hence formula (1) in the previous paragraph takes the form

$$(A\alpha)(\eta)(p) = \begin{cases} \bigl((\alpha \circ h^{-1}|A)\cdot\eta\bigr)(p) & \text{for } p \in A \\ 0 & \text{for } p \in E-A. \end{cases}$$

If we put $p = h(x)$ (where $x \in M$) and $U = h^{-1}[A]$, then

$$(A\alpha)(\eta)\bigl(h(x)\bigr) = \begin{cases} \bigl(\alpha\cdot(\eta \circ h|U)\bigr)(x) & \text{for } x \in U \\ 0 & \text{for } x \in M-U. \end{cases}$$

According to (4) § 1 we have

$$\breve{\beta}(p) = p(\beta) = \beta\left(h^{-1}(p)\right), \quad \text{for } p \in E, \beta \in C,$$

hence

$$\breve{\beta} = \beta \circ h^{-1}.$$

In the differential space we have

$$\breve{\beta}\left(h(x)\right) = h(x)(\beta) = \beta(x) \quad \text{for } x = h^{-1}(p) \in M,$$

that is

$$\beta = \breve{\beta} \circ h.$$

Now we can prove the following theorem.

T h e o r e m 2.1. If $\operatorname{supp}\alpha \subset U = h^{-1}[U]$ and $U \in \tau_C$, then the function α is subject to the set A ($\operatorname{supp}\alpha$ denotes the support of the function α).

P r o o f. Let $\eta \in C_A$. The function β defined by the formula

$$\beta(x) = (A\alpha)(\eta)\left(h(x)\right) = \begin{cases} \alpha(x) \cdot \eta\left(h(x)\right) & \text{for } x \in U, \\ 0 & \text{for } x \in M-U \end{cases}$$

belongs to C. In fact, since $\eta \circ h|U \in C_U$, there exists an open neighbourhood $V \subset U$ of the point x and a function $\gamma \in C$ such that

$$\gamma|V = (\eta \circ h|U)|V = \eta \circ h|V.$$

Hence for $y \in V$ we have $\beta(y) = \alpha(y) \cdot \gamma(y) = (\alpha \cdot \gamma)(y)$. This implies $\beta|V = (\alpha \cdot \gamma)|V$, where $\alpha \cdot \gamma \in C$. Now if $x \in M - U$, then $x \notin \operatorname{supp}\alpha = \overline{\operatorname{supp}\alpha}$. Hence there exists an open neighbourhood W of the point x, disjoint with the set $\operatorname{supp}\alpha$; that is, $\beta|W = o|W$, where o is the function identically equal to 0 on M. Thus the function β is a local C-function; that is, $\beta \in C_M = C$ (see [3]). Moreover we have

$$\breve{\beta}(p) = (\beta \circ h^{-1})(p) = (A\alpha)(\eta)(p) \quad \text{for} \quad p = h(x) \in A.$$

Consequently $\breve{\beta} = (A\alpha)(\eta)$. If we also have $\beta_1 = (A\alpha)(\eta)$, then $\breve{\beta} = \breve{\beta}_1$. This implies $\beta \circ h^{-1} = \beta_1 \circ h^{-1}$ i.e. $\beta = \beta_1$. In this way the uniqueness of the choice of the function β is proved.

Now assume that a function $\varphi : E \to R$ has the property that there exists a function $\beta \in C$ such that $\breve{\beta} = \varphi$, i.e. $\varphi = \beta \circ h^{-1}$. Moreover assume that there exists another function $\beta' \in C$ such that $\varphi = \beta' \circ h^{-1}$. Then $\beta \circ h^{-1} = \beta' \circ h^{-1}$, i.e. $\beta = \beta'$. Hence we have proved that the function β is unique. Accordingly,

$$\breve{C}(E) = \left\{\varphi ;\ \varphi : E \to R \wedge \varphi \circ h \in C\right\} \subset F(E,R)$$

Similarly we can define the set $C(E)$ of all functions of the form $[\varphi]$, where $\varphi \in \breve{C}(E)$. From the definition of $[\varphi]$ it follows that $[\varphi] = \varphi \circ h$.

Hence

$$C(E) = \left\{\varphi \circ h;\ \varphi \in \breve{C}(E)\right\} = C.$$

Incidentally we can notice that the definition of the set $C(E)$ implies that $C(E)$ is closed in the algebra $F(E,R)$ with respect to the operation of addition, multiplication, and multiplication of functions by scalars.

Now we shall formulate the basic theorem of this section.

T h e o r e m 2.2. If the topological space (M, τ_C) is paracompact and C-normal, and if a family $\mathfrak{A} \subset 2^E$ has the property that the family $\mathcal{B} = \left\{h^{-1}[A];\ A \in \mathfrak{A}\right\}$ is an open covering of the set M, then there exists a smooth decomposition of unity subject to the family $\mathfrak{A}$.

P r o o f. Let $\mathcal{B} = \left\{B_s :\ s \in S\right\}$. If the topological space is paracompact, there exists an open covering $\mathfrak{W} = \left\{V_s ;\ s \in S\right\}$ of the set contained in $\mathcal{B}$ and locally finite (hence pointwise finite - see [4],[5]). The topological space (M, τ_C) is normal as well, so that the covering $\mathfrak{W}$ contains an open subcovering $\mathcal{W}' = \{W'_s ; s \in S\}$ of the set M, which in turn contains an open subcovering $\mathcal{W} = \{W_s : s \in S\}$ such that

$$\bar{W}_s \subset W'_s \subset \bar{W}'_s \subset V_s .$$

For any $s \in S$ the sets $\bar{W}_s$ and $M - W'_s$ are disjoint and closed, hence there exists a function $f_s \in C$ such that $f_s \geqslant 0$, $f_s(x) = 1$ for $x \in \bar{W}_s$ and $f_s(x) = 0$ for $x \in M - W'_s$. Therefore $\operatorname{supp} f_s \subset \bar{W}'_s \subset V_s$. The sum $\sum_{s \in S} f_s$ is well defined ($\mathcal{V}$ is a locally finite family) and belongs to C. Moreover, $\sum_{s \in S} f_s(x) > 0$ for $x \in M$, since $\mathcal{W}$ is a covering of the set M.

The family $(\varphi_s : s \in S)$ of functions of the form

$$\varphi_s = f_s \Big/ \sum_{s \in S} f_s \quad \text{for} \quad s \in S$$

is a smooth decomposition of unity subject to the family $\mathfrak{A}$.

In fact, since $\mathcal{B} = \{h^{-1}[A] : A \in \mathfrak{A}\} = \{B_s : s \in S\}$, we infer that

(i) $$\bigcup_{s \in S} \{p;\ p \in A_s \wedge p(\varphi_s) \neq 0\} = \bigcup_{s \in S} \{p;\ h^{-1}(p) \in B_s \wedge \varphi_s(h^{-1}(p)) \neq 0\} \supset$$

$$\supset \bigcup_{s \in S} \{p;\ h^{-1}(p) \in W_s\} \supset E,$$

because $\mathcal{W}$ is a covering of the set M.

(ii) Let $p \in E$. Since $\operatorname{supp} \varphi_s \subset V_s$ for $s \in S$, and the family $\mathcal{V}$ is locally finite, it is pointwise finite and $\{s : p(\varphi_s) \neq 0\} = \{s : \varphi_s(h^{-1}(p)) \neq 0\}$ is a finite set.

(iii) Let $s \in S$. Since $\operatorname{supp} \varphi_s \subset V_s \subset B_s = h^{-1}[A_s]$, the function φ_s is subject to the set A_s.

Consider an arbitrary indexed family $(\eta_s : s \in S) \in \mathbb{P}_{s \in S} C_{A_s}$. For every $s \in S$ there exists exactly one function $\beta_s \in C$ such that $\check{\beta}_s = (A_s \varphi_s)(\eta_s)$. Moreover $\operatorname{supp} \beta_s \subset \operatorname{supp} \varphi_s \cap \operatorname{supp}(\eta_s \circ h | h^{-1}[A]) \subset V_s$. As the family $\mathcal{V}$ is locally finite, for any $x \in M$ there exists a neighbourhood U such that the set of those indices $s \in S$, for which $\beta_s(y) \neq 0$ for some $y \in U$, is finite. Suppose that this set is $\{s_1, \ldots, s_k\}$. Let us put

$$\beta(z) = \sum_{s \in S} \beta_s(z) = \sum_{i=1}^{k} \beta_{s_i}(z), \quad \text{for } z \in U. \tag{7}$$

Hence the function β is a local C-function on the set M, i.e. $\beta \in C$. The function β is uniquely determined. In fact, let β_1 as well satisfy equation (7). Then

$$\beta_1 = \sum_{s \in S} \beta_s, \quad \text{i.e.} \quad \breve{\beta}_1 = \beta \circ h^{-1} = \sum_{s \in S} \beta_s \circ h^{-1}.$$

But

$$\breve{\beta} = \sum_{s \in S} \breve{\beta}_s = \sum_{s \in S} \beta_s \circ h^{-1}.$$

This implies $\breve{\beta}_1 = \breve{\beta}$, and $\beta \circ h^{-1} \circ h = \breve{\beta}_1 \circ h^{-1} \circ h$. Hence $\beta = \beta_1$.

Condition (iv) follows directly from the definitions of the functions f_s and φ_s.

(v) $$\sum_{s \in S} p(\varphi_s) = \sum_{s \in S} \varphi_s\left(h^{-1}(p)\right) =$$

$$= \sum_{s \in S} \left(f_s\left(h^{-1}(p)\right) \Big/ \sum_{s \in S} f_s\left(h^{-1}(p)\right)\right) = 1,$$

since for any $p \in E$ the above sums have a finite number of components.

3. Scalar product on a C-module of vector fields

Let V be a C-module, where C is an arbitrary associative R-algebra (see Chapter I, § 1). By a scalar product on V we understand every bilinear map

$$g: V \times V \longrightarrow C \tag{8}$$

satisfying the conditions

(i) $g(X,Y) = g(Y,X)$ for $X,Y \in V$

(ii) the map $X \longmapsto g(X,\cdot)$ is an isomorphism of V onto V^*.

The bilinear map (8) satisfying condition (i) only, is called a generalized scalar product on V.

A generalized scalar product g on Diff C is said to be ε-positive definite (briefly: positive), where C is an R-algebra and ε - an orientation of the ring R, if for any $p \in E$ we have

$$p\big(g(X,X)\big) > 0,$$

where $p\big(X(\alpha)\big) \neq 0$ for some $\alpha \in C$.

Similarly, a generalized scalar product g on Diff C_A is said to be ε-positive definite, if for any $p \in A$ we have

$$p_A\big(g(X,X)\big) > 0,$$

where $p_A\big(X(\alpha)\big) \neq 0$, for some $\alpha \in C_A$.

T h e o r e m 3.1. If g is a scalar product on Diff C_A, where $A \in (C)$ (see Chpt.I § 1) and $\breve{C}(E)$ is a closed set in the algebra $F(E,R)$, then for every function α subject to the set A, the function

$$(9) \qquad (X,Y) \longrightarrow \Big[(A\alpha)\Big(g\big(A'(X),A'(Y)\big)\Big)\Big]$$

is a generalized scalar product on Diff C.

P r o o f. The additivity of the map (9) with respect to both variables is obvious. From condition (i) it follows that the map (9) satisfies the same condition.

To show that (9) is homogeneous observe first that from formula (3) § 1 of this chapter and from formulas (1) and (4) § 1, Chpt.I it follows that the following identity holds:

$$(10) \qquad (A\alpha)\big(\hat{A}(\beta)\big) = \breve{\beta}\cdot(A\alpha)(1_A) \quad \text{for } \alpha,\beta \in C.$$

Hence by formula (2), § 3 Chpt.I we have

$$(A\alpha)\big(g(A'(\beta\cdot X),A'(Y))\big) = (A\alpha)\big(\hat{A}(\beta)\cdot g(A'(X),A'(Y))\big) =$$

$$= (A\alpha)\big(\hat{A}(\beta)\big)\cdot(A\alpha)\ g\big(A'(X),A'(Y)\big) = \breve{\beta}\cdot(A\alpha)(1_A)\cdot(A\alpha)\big(g(A'(X)\ ,$$

$$A'(Y)\big) = \breve{\beta}\,(A\alpha)\big(1_A\cdot g(A'(X),\ A'(Y))\big) = \breve{\beta}\cdot(A\alpha)\big(g(A'(X),A'(Y))\big)\ .$$

According to Theorem 1.1 we have

$$\big[(A\alpha)\big(g(A'(\beta\cdot X),A'(Y))\big)\big] = \big[\breve{\beta}\big]\ \big[(A\alpha)\big(g(A'(X),A'(Y))\big)\big] =$$

$$= \beta\cdot\big[(A\alpha)\big(g(A'(X),A'(Y))\big)\big]\ .$$

Now we shall formulate a theorem of basic importance for the construction of scalar product on the C-module Diff C.

T h e o r e m 3.2. If $\mathfrak{A} \subset (C)$, $E \subset \mathrm{Hom}\,(C,R)$, φ_s: $s \in S$ is a smooth decomposition of unity subject to the family $\mathfrak{A}$, and if the function $s \longmapsto A_s$ is a choice function for this decomposition, then the correspondence

$$(X,Y) \to \Big[\sum_{s\in S}(A_s\varphi_s)\Big(g_{A_s}\big(A_s'(X),A_s'(Y)\big)\Big)\Big] \tag{11}$$

is a positive definite scalar product on Diff C, under the assumption that g_A is a positive definite scalar product on Diff C_{A_s} for $s \in S$.

P r o o f. If $s \in S$, and $X,Y \in \mathrm{Diff}\ C$, then $A_s'(X)$, $A_s'(Y) \in \mathrm{Diff}\ C_A$. Hence $\eta_s = g_{A_s}\big(A_s'(X),\ A_s'(Y)\big)$ belongs to C_{A_s}. According to condition (iii) § 1 appearing in the definition of smooth decomposition of unity, there exists exactly one element $\beta \in C$ such that the equality (6), § 1 holds. Thus we have

$$\beta = \Big[\sum_{s\in S}(A_s\varphi_s)\Big(g_{A_s}\big(A_s'(X),\ A_s'(Y)\big)\Big)\Big]\ .$$

From Theorem 3.1. it follows that the function

$$(X,Y) \longrightarrow \left[(A_s\varphi_s)\left(g_{A_s}\left(A'_s(X),A'_s(Y)\right)\right)\right],$$

is a generalized scalar product on Diff C. From Theorem 1.1 it follows that the bilinear map (4) is a generalized scalar product on Diff C. It remains to show that the map (4) is positive definite.

Let us take any $p \in E$ and let $p\big(X(\alpha)\big) \neq 0$ for some $\alpha \in C$. From condition (ii) of the definition of smooth decomposition of unity it follows that the set $\left\{s\colon o \in A_s \wedge p(\varphi_s) \neq 0\right\}$ is finite.

Suppose that this set is $\left\{s_1,\dots,s_k\right\}$. Let $\eta_{s_i} = g_{A_{s_i}}\left(A'_{s_i}(X),A'_{s_i}(X)\right)$, for $i=1,\dots,k$. Since $p\big(X(\alpha)\big) \neq 0$ and

$$p\big(X(\alpha)\big) = p_{A_{s_i}}\left(\hat{A}(X\alpha)\right) = p_{A_{s_i}}\left(A'_{s_i}(X)\left(\hat{A}_{s_i}(\alpha)\right)\right) \neq 0.$$

and since the scalar product $g_{A_{s_i}}$ is positive definite on Diff $C_{A_{s_i}}$ by assumption, we obtain $p_{A_{s_i}}\left(\eta_{s_i}\right) > 0$ for $i=1,\dots,k$, This implies

$$p\left(\left[\sum_{s\in S}(A_s\varphi_s)\left(g_{A_s}\left(A'_s(X),A_s(X)\right)\right)\right]\right) =$$

$$= \left(\sum_{s\in S}(A_s\varphi_s)\left(g_{A_s}\left(A'_s(X),A'_s(X)\right)\right)(p) =\right.$$

$$= \sum_{i=1}^{k}\left(\left(A_{s_i}\varphi_{s_i}\right)\left(\eta_{s_i}\right)(p)\right) = \sum_{i=1}^{k} p\left(\varphi_{s_i}\right)\cdot p_{A_{s_i}}\left(\eta_{s_i}\right) > 0,$$

because $p\left(\varphi_{s_i}\right) > 0$ for $i=1,\dots,k$, by the definition of the set $\left\{s_1,\dots,s_k\right\}$ and by condition (iv) of the definition of smooth decomposition of unity. This concludes the proof of the theorem.

T h e o r e m 3.3. If there is a family $\mathfrak{A} \subset (C)$ (where $C = C(E)$, $E \subset \mathrm{Hom}\,(C,R)$, (R,ε) is an oriented ring) such that there exists a smooth decomposition of unity subject to the family $\mathfrak{A}$, and if for every $A \in \mathfrak{A}$ there exists a positive definite scalar product on Diff C_A, then there exists a positive definite scalar product on Diff C.

If, moreover, the R-algebra C has the property that every positive definite scalar product on Diff C is a scalar product, and if the ring R is diadic, then there exists a covariant derivative on Diff C. (The ring R is said to be diadic if for every $r \in R$ there is exactly one element $p \in R$ such that $r = p+p$).

P r o o f. The first part of the theorem is a direct consequence of Theorem 3.2. The second part follows from the first by Theorem 9.4 of paper [1].

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DEPARTMENT OF MATHEMATICS, PHYSICS AND CHEMISTRY, UNIVERSITY OF ŁÓDŹ

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