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SOME PROPERTIES OF LINEAR CONNECTIONS ON RIEMANNIAN MANIFOLDS, II.

Let M be a smooth n-dimensional Riemannian manifold with a metric tensor $g = \langle , \rangle$. FM denotes the ring of smooth functions on M and BM denotes the FM-module of smooth vector fields on M.

It is known that if a linear connection ∇ on M is Riemannian, then by definition we have

(1) $$\bigwedge_{X,Y,Z\in BM} Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle .$$

If, moreover,

(2) $$\bigwedge_{X,Y\in BM} T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$$

then this Riemannian connection is called a Levi-Civita connection*).

Let $f \in FM$ be an arbitrary smooth function on M.

D e f i n i t i o n 1.1. The vector field, denoted by ∇f, defined by the formula

(3) $$\bigwedge_{X\in BM} \langle \nabla f, X\rangle = X(f) = df(x),$$

where df is the differential of the function f, is called the gradient of the function f. The tensor field h_f : $BM \times BM \longrightarrow FM$ defined by the formula

*) D.Gromoll, W.Klingenberg, W.Meyer: Riemannsche Geometrie in Grossen. Berlin 1968.

$$(4) \qquad h_f(X,Y) = \langle \nabla_X \nabla f, Y \rangle$$

is called a Hess 2-form.

We shall prove the following theorem.

Theorem 1. A Riemannian connection ∇ on a Riemannian manifold M is a Levi-Civita connection on M if and only if for any function $f \in FM$ the Hess 2-form is a symmetric 2-form.

Proof. Let ∇ be a Riemannian connection on M and f an arbitrary function in FM. For any $X,Y \in BM$ we then have

$$\begin{cases} X\langle \nabla f, Y\rangle = \langle \nabla_X \nabla f, Y\rangle + \langle \nabla f, \nabla_X Y\rangle, \\ Y\langle \nabla f, X\rangle = \langle \nabla_Y \nabla f, X\rangle + \langle \nabla f, \nabla_Y X\rangle. \end{cases}$$

From the definition (4) of a Hess 2-form we obtain

$$(5) \qquad \begin{aligned} h_f(X,Y) - h_f(Y,X) &= \langle \nabla_X \nabla f, Y\rangle - \langle \nabla_Y \nabla f, X\rangle = \\ &= X\langle \nabla f, Y\rangle - Y\langle \nabla f, X\rangle - \langle \nabla f, \nabla_X Y\rangle + \langle \nabla f, \nabla_Y X\rangle = \\ &= [X,Y](f) - \langle \nabla f, T(X,Y) + [X,Y]\rangle = (T(X,Y))(f), \end{aligned}$$

which shows that ∇ is a Levi-Civita connection on M.

Now assume that ∇ is a Levi-Civita connection on M. Then for all $X,Y \in BM$ $T(X,Y) = 0$, and consequently from (5) it follows that $h_f(X,Y) = h_f(Y,X)$ for $f \in FM$. The remaining part of the proof is evident.

Let L be a tensor field on M of type (2,1), that is $L : BM \times BM \longrightarrow BM$.

Definition 1.2. A tensor field $L : BM \times BM \longrightarrow BM$ on a Riemannian manifold M is said to be right orthogonal provided that

$$(6) \qquad \bigwedge_{X,Y \in BM} \langle X, L(Y,X)\rangle = 0,$$

it is said to be left orthogonal if

$$\bigwedge_{X,Y\in BM} \langle X, L(X,Y)\rangle = 0. \tag{7}$$

When L is left and right orthogonal we say that L is orthogonal.

We shall prove the following theorem.

T h e o r e m 2. A tensor field $L : BM \times BM \longrightarrow BM$ on a Riemannian manifold M is right orthogonal if and only if

$$\langle Y, L(Z,X)\rangle + \langle X, L(Z,Y)\rangle = 0. \tag{8}$$

Similarly, L is left orthogonal if and only if

$$\langle Y, L(X,Z)\rangle + \langle X, L(Y,Z)\rangle = 0. \tag{9}$$

P r o o f. Assume that L is a tensor field on M which is right orthogonal. By definition we have

$$\bigwedge_{\tilde{X},Z\in BM} \langle \tilde{X}, L(Z,\tilde{X})\rangle = 0.$$

Putting $\tilde{X} = X + Y$, $X, Y \in BM$, we get

$$\langle X + Y, L(Z, X+Y)\rangle = 0.$$

This implies (8). The remaining part of the proof is obvious.

Theorem 2 directly implies the following corollary.

C o r o l l a r y 1. An orthogonal tensor field $L : BM \times BM \longrightarrow BM$ on M is skew-symmetric, that is we have

$$\bigwedge_{X\in BM} L(X,X) = 0.$$

Next we prove the following theorem.

T h e o r e m 3. Let $\tilde{\nabla}$ be a linear connection on a Riemannian manifold M and let ∇ be a Levi-Civita connection on M. $\tilde{\nabla}$ is a Riemannian connection if and only if $S = \nabla - \tilde{\nabla}$ is a right orthogonal tensor field on M.

P r o o f. Assume that $S = \nabla - \tilde{\nabla}$ is a right orthogonal tensor field on M of type (2,1). Then we have

$$\bigwedge_{X,Y,Z\in BM} Z\langle X,Y\rangle = \langle\nabla_Z X,Y\rangle + \langle X, \nabla_Z Y\rangle \tag{10}$$

and

$$\bigwedge_{X,Y,Z\in BM} \langle X,S(Z,Y)\rangle + \langle Y,S(Z,X)\rangle = 0. \tag{11}$$

From the assumption $\nabla = S + \tilde{\nabla}$ and from (10) we obtain

$$\bigwedge_{X,Y,Z\in BM} Z\langle X,Y\rangle = \langle S(Z,X) + \tilde{\nabla}_Z X,Y\rangle + \langle X,S(Z,Y) + \tilde{\nabla}_Z Y\rangle =$$

$$= \langle\tilde{\nabla}_Z X,Y\rangle + \langle X, \tilde{\nabla}_Z Y\rangle + \langle Y,S(Z,X)\rangle + \langle X,S(Z,Y)\rangle.$$

By (11) this implies

$$Z\langle X,Y\rangle = \langle\tilde{\nabla}_Z X,Y\rangle + \langle X, \tilde{\nabla}_Z Y\rangle ; \quad X,Y,Z \in BM.$$

This shows that $\tilde{\nabla}$ is a Riemannian connection.

Conversely, assume that ∇ and $\tilde{\nabla}$ are Riemannian connections on M. Then

$$\bigwedge_{X,Y,Z\in BM} Z\langle X,Y\rangle = \langle\nabla_Z X,Y\rangle + \langle X,\nabla_Z Y\rangle \wedge Z\langle X,Y\rangle =$$

$$= \langle\tilde{\nabla}_Z X,Y\rangle + \langle X, \tilde{\nabla}_Z Y\rangle.$$

Hence for arbitrary $X,Y,Z \in BM$ we have

$$\langle\nabla_Z X - \tilde{\nabla}_Z X,Y\rangle + \langle\nabla_Z Y - \tilde{\nabla}_Z Y,X\rangle = 0.$$

Putting $S = \nabla - \tilde{\nabla}$ we obtain

$$\langle Y,S(Z,X)\rangle + \langle X,S(Z,Y)\rangle = 0.$$

This ends the proof of Theorem 3.

Theorem 3 implies the following corollary.

C o r o l l a r y 2. An arbitrary Riemannian connection on M differs from a Levi-Civita connection by a right orthogonal tensor S of type (2,1).

Let ∇ and $\tilde{\nabla}$ be arbitrary Riemannian connections on M. From Corollary 2 it follows that the tensor field $S = \nabla - \tilde{\nabla}$ is right orthogonal. Consequently, for any tensor fields $X,Y,Z \in BM$ we have

$$\langle \nabla_X Y, Z \rangle = \langle \tilde{\nabla}_X Y, Z \rangle + \langle S(X,Y), Z \rangle. \tag{12}$$

Putting $Y = Z$ in (12) we obtain

$$\langle \nabla_X Y, Y \rangle = \langle \tilde{\nabla}_X Y, Y \rangle \quad \text{for} \quad X,Y \in BM. \tag{13}$$

Let x be any chart with domain U on the manifold M. Denoting $X_i = \frac{\partial}{\partial x_i}$, $i = 1,2,\dots,n$, we obtain from (13)

$$\langle \nabla_{X_i} X_j, X_j \rangle = \langle \tilde{\nabla}_{X_i} X_j, X_j \rangle$$

or equivalently

$$\Gamma_{ij}^{k} g_{jk} = \tilde{\Gamma}_{ij}^{k} g_{jk}$$

where Γ_{ij}^{k} and $\tilde{\Gamma}_{ij}^{k}$ are the coefficients of the connections ∇ and $\tilde{\nabla}$ in the chart x. In particular this gives the following corollary.

C o r o l l a r y 3. The coefficients $\tilde{\Gamma}_{ij,j} = \tilde{\Gamma}_{ij}^{k} g_{kj}$ $(i,j = 1,2,\dots,n)$ of any Riemannian connection on M are equal to the respective coefficients of a Levi-Civita connection on M, that is, to Christoffel's symbols.

In the sequel we assume that on a Riemannian manifold M there are given two linear connections ∇ and $\tilde{\nabla}$ differing by a tensor $S : BM \times BM \longrightarrow BM$. Let x be an arbitrary chart on M with domain U. Then in the domain of the chart x we have by assumption

$$\Gamma_{ij}^{k} = \tilde{\Gamma}_{ij}^{k} + S_{ij}^{k}, \tag{14}$$

where Γ^k_{ij}, $\tilde{\Gamma}^k_{ij}$ and S^k_{ij} are, respectively, the coefficients of the connections ∇, $\tilde{\nabla}$ and of the tensor S in the chart x.

It is known that

$$(15) \qquad R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{i\varrho}\Gamma^{\varrho}_{jk} - \Gamma^l_{j\varrho}\Gamma^{\varrho}_{ik},$$

where $R_{ijk}{}^{l}$ are the coefficients of the curvature tensor of the connection ∇ in the chart x.

From (14) and (15) we obtain after simple calculation

$$(16) \quad R_{ijk}{}^{l} = \tilde{R}_{ijk}{}^{l} + 2\tilde{\nabla}_{[i} S_{j]k}{}^{l} + 2\tilde{\Gamma}_{[ij]}{}^{\varrho} S_{\varrho k}{}^{l} + 2\, S_{[i|\varrho|}{}^{l} S_{j]k}{}^{\varrho}.$$

Now, if we assume that the space under examination is a 4-dimensional Riemannian space with a Levi-Civita connection and S is an ortogonal and self-dual 1/2 tensor (cf.I part), them from (16) we get

$$(17) \qquad R_{ijkl} = \tilde{R}_{ijkl} + 2\tilde{\nabla}_{[i} S_{j]kl}.$$

Hence we obtain the following theorem.

T h e o r e m 4. If a Riemannian connection ∇ on M_4 differs from a Levi-Civita connection $\tilde{\nabla}$ on M_4 by an ortogonal and self-dual 1/2 -tensor S satisfying the identity

$$\tilde{\nabla}_{[i} S_{j]kl} = 0,$$

on the domain of an arbitrary chart x belonging to the atlas of M_4, then the curvature tensors of both connections are equal.

This theorem implies the following corollary.

Corollary 4. If the Riemannian space $(M_4, g, \tilde{\nabla})$ is a space of constant curvature, where $\tilde{\nabla}$ is a Levi-Civita connection, then the Riemannian space (M_4, q, ∇) is a space of constant curvature as well, provided that ∇ is a connection satisfying the assumptions of Theorem 4.

From (17) we obtain the identitics

(18) $$R_{jk} = \tilde{R}_{jk} + g^{il}\tilde{\nabla}_i S_{jkl}\ .$$

Hence we get

(19) $$R^*_{jk} \overset{df}{=} R_{(jk)} = \tilde{R}_{jk}.$$

Now, let us introduce the following definition.

D e f i n i t i o n 1.3. For an arbitrary connection ∇ on the manifold M the tensor R^*, satisfying the identities

$$R^*_{ij} = R_{l(ij)}{}^{l}$$

on the domain of an arbitrary chart x belonging to the atlas of M, where $R_{ijk}{}^{l}$ are the coefficients of the curvature tensor of ∇, is called the Ricci tensor of the connection ∇.

Using the above definition and identities (19) we obtain the following theorem.

T h e o r e m 5. If a Riemannian connection ∇ on M_4 differs from a Levi-Civita connection $\tilde{\nabla}$ on M_4 by an ortogonal and self-dual 1/2 -tensor S, then the Ricci tensors of both connections are equal.

We now introduce the following definition.

D e f i n i t i o n 1.4. A Riemannian space M with an arbitrary connection ∇ is called a generalized Finstein space if

$$R^*_{ij} = \lambda g_{ij}$$

where λ is a scalar factor.

Consequently, from Theorem 5 we get

T h e o r e m 6. If the Riemannian space $(M_4, g, \tilde{\nabla})$ is an Einstein space, where $\tilde{\nabla}$ is a Levi-Civita connection, then the Riemannian space (M_4, g, ∇) is a generalized Einstein space (in the sense of definition 1.4) provided that ∇ is a connection satisfying the assumptions of Theorem 5.

Of course in general the Einstein space is a space without torsion but the generalized Einstein space is a space with torsion.

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