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Marian Jaszczak, Zbigniew Żekanowski

SOME PROPERTIES OF LINEAR CONNECTIONS ON RIEMANNIAN MANIFOLDS, I.

1. Introduction

Let $M = (M,A)$ be an n-dimensional Riemannian manifold of class C^{∞}, A - the atlas of this manifold.

FM and BM denote, respectively, the ring of smooth functions on M, and the module of smooth vector fields on M. Further, let g denote the metric tensor and ∇ - an arbitrary linear connection on M.

D e f i n i t i o n 1.1. Every tensor field $\omega: BM^r \longrightarrow FM$ satisfying, in the domain of an arbitrary chart $x = (x^1,\ldots,x^n) \in A$, the identities

$$(1) \qquad \omega_{[i_1,\ldots,i_r]} = \omega_{i_1,\ldots,i_r}$$

$$(2) \qquad \nabla_{[i}\omega_{i_1,\ldots,i_r]} = 0$$

$$(3) \qquad g^{ij}\nabla_i\,\omega_{j,i_2,\ldots i_r} = 0$$

is called a field of harmonic tensors [1].

D e f i n i t i o n 1.2. Every tensor field $\omega: BM^r \longrightarrow FM$ satisfying, on an arbitrary chart $x \in A$, the identities (1) and (3) as well as

$$(4) \qquad \nabla_{[i}\omega_{i_1,\ldots,i_r]} = \nabla_i\omega_{i_1,\ldots,i_r}$$

is called a field of Killing tensors [1].

D e f i n i t i o n 1.3. Let $n = 2p$. A tensor field $\omega : BM^{p+1} \to FM$ is called a field of self-dual 1/p-tensors if in the domain of an arbitrary chart $x \in A$ the following identities hold:

(5) $$\omega_{i[i_1,\dots,i_p]} = \omega_{ii_1,\dots,i_p}$$

(6) $$\omega_{ii_1,\dots,i_p} = \frac{\theta}{p!} I_{i_1,\dots,i_p}{}^{j_1,\dots,j_p} \omega_{ij_1,\dots,j_p}$$

where $I : BM^n \to FM$ is a linear n-form on M such that

$$I^2 = I_{i_1,\dots,i_p}{}^{i_1,\dots,i_p} = 1; \qquad \theta = \pm 1.$$

A tensor field $\underset{1}{\omega}$ is said to be self-dual of the first kind if $\theta = +1$. Similarly, a tensor field $\underset{2}{\omega}$ is said to be self-dual of the second kind if $\theta = -1$ [2].

For self-dual 1/p-tensors the following theorem holds (see [2]).

T h e o r e m 1.1. For two arbitrary self-dual 1/p-tensors $\underset{1}{\omega}$ and $\underset{2}{\omega}$, of different kinds if $p = 2k$, and of the same kind if $p = 2k + 1$, the identities

(7) $$\underset{1}{\omega}_{ii_1,\dots,i_{p-1}[j} \; \underset{2}{\omega}_{|k|}{}^{i_1,\dots,i_{p-1}}{}_{l]} = 0$$

hold in any chart $x \in A$.

2. <u>Some properties of linear connections on Riemannian manifolds</u>

Let M be a four-dimensional Riemannian manifold, x - an arbitrary chart in the atlas of this manifold. Next, let $S_{\alpha ij}$ be the coordinates, in the chart x, of any self-dual 1/p-tensor S. By definition, the following identities hold:

$$S_{\alpha[ij]} = S_{\alpha ij}, \quad S_{\alpha ij} = \frac{\theta}{2} I_{ijkl} S_{\alpha}^{kl}$$

where $\theta = +1$ or -1.

We shall prove the following theorem.

T h e o r e m 2.1. Two linear connections ∇ and $\widetilde{\nabla}$ on the Riemannian manifold M_4 differ by a self-dual 1/p-tensor S of the first kind (resp. of the second kind) if and only if for any self-dual bi-vector ω of the second kind (resp. of the first kind) the following identity holds

$$\nabla\omega = \widetilde{\nabla}\omega .$$

P r o o f. Let $S = \nabla - \widetilde{\nabla}$ where ∇ and $\widetilde{\nabla}$ are linear connections on M_4. Then for any chart x from the atlas of the manifold M_4 we have the identities

$$S_{ij}^{\ k} = \Gamma_{ij}^{\ k} - \widetilde{\Gamma}_{ij}^{\ k}$$

where $S_{ij}^{\ k}$, $\Gamma_{ij}^{\ k}$ and $\widetilde{\Gamma}_{ij}^{\ k}$ are, respectively, the coordinates of the tensor S, and the coefficients of the connections ∇ and $\widetilde{\nabla}$ in the chart x.

Now let ω_{ij} be the coordinates of an arbitrary self-dual bi-vector. By definition we have

$$(8) \qquad \begin{cases} \nabla_{\alpha}\omega_{ij} = \partial_{\alpha}\omega_{ij} - \Gamma_{\alpha i}^{\varrho}\omega_{\varrho j} - \Gamma_{\alpha j}^{\varrho}\,\omega_{i\varrho} \\ \widetilde{\nabla}_{\alpha}\omega_{ij} = \partial_{\alpha}\omega_{ij} - \widetilde{\Gamma}_{\alpha i}^{\varrho}\omega_{\varrho j} - \widetilde{\Gamma}_{\alpha j}^{\varrho}\,\omega_{i\varrho} \end{cases} .$$

From the identity (8) we obtain

$$\nabla_{\alpha}\omega_{ij} - \widetilde{\nabla}_{\alpha}\,\omega_{ij} = S_{\alpha i}^{\ \varrho}\,\omega_{j\varrho} - S_{\alpha j}^{\ \varrho}\,\omega_{i\varrho}$$

or equivalently

$$(9) \qquad \nabla_{\alpha}\omega_{ij} - \widetilde{\nabla}_{\alpha}\,\omega_{ij} = 2S_{\alpha[i}^{\ \ \varrho}\,\omega_{j]\varrho} .$$

By Theorem 1.1 this implies that $S_{\alpha[i}{}^{\varrho}\,\omega_{j]\varrho} = 0 \Longleftrightarrow S_{\alpha i}{}^{\varrho}$ are the coordinates of any self-dual 1/2-tensor of the opposite kind to that of the self-dual bi-vector ω (see [2]).

Theorem 2.1 implies the following corollaries.

C o r o l l a r y 2.2. If two linear connections ∇ and $\widetilde{\nabla}$ on the Riemannian manifold M_4 differ by a self/dual 1/2-tensor S, then every field of self-dual bi-vectors ω of the opposite kind to S is a covariant constant field with respect to the connection ∇ whenever it is a covariant constant field with respect to the connection $\widetilde{\nabla}$.

C o r o l l a r y 2.3. If two linear connections ∇ and $\widetilde{\nabla}$ on the Riemannian manifold M_4 differ by a self-dual 1/2--tensor S, then every field of self-dual bi-vectors ω of the opposite kind to S is harmonic with respect to the connection ∇ whenever it is harmonic with respect to the connection $\widetilde{\nabla}$.

Now let M_n be an arbitrary n-dimensional Riemannian manifold of class C^{∞}, and let ∇ and $\widetilde{\nabla}$ be two linear connections on M_n. Further, let $S: BM^3 \longrightarrow FM$ be a field symmetric with respect to its arguments satisfying, within the domain of an arbitrary chart $x \in M$, the identities

$$g^{ij}\, S_{ijk} = 0. \tag{10}$$

Let us put

$$\Gamma_{ij}^{k} - \widetilde{\Gamma}_{ij}^{k} = S_{ijl}\, g^{lk} = S_{ij}{}^{k}. \tag{11}$$

Consider now any antisymmetric tensor field $\omega: BM^p \longrightarrow FM$. Then, by definition, in the domain of the chart x the following identities hold.

$$(12)\begin{cases} \nabla_{\alpha}\omega_{i_1,\dots,i_p} = \partial_{\alpha}\omega_{i_1,\dots,i_p} - \sum\limits_{l=1}^{p} \Gamma_{\alpha i_l}^{\varrho}\, \omega_{i_1,\dots,i_{l-1},\varrho,i_{l+1},\dots,i_p} \\ \widetilde{\nabla}_{\alpha}\omega_{i_1,\dots,i_p} = \partial_{\alpha}\omega_{i_1,\dots,i_p} - \sum\limits_{l=1}^{p} \widetilde{\Gamma}_{\alpha i_l}^{\varrho}\, \omega_{i_1,\dots,i_{l-1},\varrho,i_{l+1},\dots,i_p} \end{cases}$$

Taking into account (11) we obtain from (12)

$$(13)\quad \nabla_\alpha \omega_{i_1,\dots,i_p} - \tilde{\nabla}_\alpha \omega_{i_1,\dots,i_p} = -\sum_{l=1}^{p} S_{\alpha i_l}^{\ \ \varrho}\, \omega_{i_1,\dots,i_{l-1},\varrho,i_{l+1},\dots,i_p}.$$

In view of the assumptions concerning the tensor S, we obtain from (13) the identities

$$(14)\quad \nabla_{[\alpha}\omega_{i_1,\dots,i_p]} = \tilde{\nabla}_{[\alpha}\omega_{i_1,\dots,i_p]}$$

and

$$(15)\quad g^{\alpha i}\nabla_\alpha \omega_{ii_2,\dots,i_p} = g^{\alpha i}\tilde{\nabla}_\alpha \omega_{ii_2,\dots,i_p}.$$

From (14) and (15) we obtain the following theorem.

T h e o r e m 2.3. If two linear connections ∇ and $\tilde{\nabla}$ on the Riemannian manifold M_n differ by a tensor S satisfying, in the domain of any chart x from the atlas of the manifold M_n, the identities

$$(16)\quad \begin{cases} S_{(ijk)} = S_{ijk}, \\ g^{ij}\, S_{ijk} = 0 \end{cases}$$

then the rotation as well as the divergence of any p-vector with respect to both connections are the same.

In particular, the above theorem implies the following corollaries.

C o r o l l a r y 2.4. Assume that two linear connections ∇ and $\tilde{\nabla}$ on the Riemannian manifold M_n differ by a tensor S satisfying the identities (16) within the domain of an arbitrary chart x from the atlas of the manifold M_n. Then

1) Every tensor field on M_n harmonic with respect to the connection ∇ is also harmonic with respect to the connection $\tilde{\nabla}$.

2) Every field of Killing tensors on M_n with respect to the linear connection ∇ is also a field of harmonic tensors with respect to the connection $\tilde{\nabla}$.

It is not difficult to see that the following theorem also holds.

T h e o r e m 2.5. If two linear connections ∇ and $\tilde{\nabla}$ on the Riemannian manifold M_n differ by a tensor S satisfying, within the domain of a chart x from the atlas of the manifold M_n, the identities

$$S_{(ijk)} = S_{ijk},$$

then every field of Killing tensors with respect to the linear connection ∇ is also a field of Killing tensors with respect to the connection $\tilde{\nabla}$.

It is known that the torsion tensors T and $\tilde{T}$ of two linear connections on a differential manifold M_n are identical if and only if the connections ∇ and $\tilde{\nabla}$ determining these tensors differ by a symmetric tensor.

Accordingly, (2.5) implies the following corollary.

C o r o l l a r y 2.6. If the torsion tensors T and $\tilde{T}$ of two linear connections ∇ and $\tilde{\nabla}$ on the differential manifold M_n are equal, then the rotation of an arbitrary tensor $\omega : BM^p \rightarrow FM$ with respect to the connection ∇ equals the rotation of this tensor with respect to the connection $\tilde{\nabla}$.

In particular, we obtain the following corollary.

C o r o l l a r y 2.7. If the torsion tensors T and $\tilde{T}$ of two linear connections ∇ and $\tilde{\nabla}$ on the Riemannian manifold M_n are equal, then every Killing tensor ω with respect to the connection ∇ is simultaneously a Killing tensor with respect to the connection $\tilde{\nabla}$.

In turn we consider a tensor field $S : BM^2 \rightarrow BM$ satisfying the identities

$$(17) \qquad \nabla_{[\alpha} S_{i]j}{}^{k} = 0; \quad S_{ij}{}^{k} = - S_{ji}{}^{k}; \quad g^{ij} \nabla_i S_{jk}{}^{l} = 0$$

in the domain of a chart x from the atlas of the manifold M_n. Then we have

$$\begin{cases} \nabla_\alpha S_{ij}^{\ \ k} = \partial_\alpha S_{ij}^{\ \ k} + \Gamma_{\alpha\varrho}^{k} S_{ij}^{\varrho} - \Gamma_{\alpha j}^{\varrho} S_{i\varrho}^{\ \ k} - \Gamma_{\alpha i}^{\varrho} S_{\varrho j}^{\ \ k} \\ \tilde{\nabla}_\alpha S_{ij}^{\ \ k} = \partial_\alpha S_{ij}^{\ \ k} + \tilde{\Gamma}_{\alpha\varrho}^{k} S_{ij}^{\varrho} - \tilde{\Gamma}_{\alpha j}^{\varrho} S_{i\varrho}^{\ \ k} - \tilde{\Gamma}_{\alpha i}^{\varrho} S_{\varrho j}^{\ \ k}. \end{cases}$$

Hence we get

$$\nabla_{(\alpha}S_{i)j}^{\ \ k} - \tilde{\nabla}_{(\alpha}S_{i)j}^{\ \ k} = S_{(\alpha|\varrho|}^{\ \ \ \ k} S_{i)j}^{\ \ \varrho} - S_{(\alpha|j|}^{\ \ \ \ \varrho} S_{i)\varrho}^{\ \ k} - S_{(\alpha i)}^{\ \ \ \varrho} S_{\varrho j}^{\ \ k} =$$

$$= S_{(\alpha|\varrho|}^{\ \ \ \ k} S_{i)j}^{\ \ \varrho} - S_{(i|j|}^{\ \ \ \ \varrho} S_{\alpha)\varrho}^{\ \ k} - S_{(\alpha i)}^{\ \ \ \varrho} S_{\varrho j}^{\ \ k} = 0.$$

Consequently, we obtain the following corollary.

C o r o l l a r y 2.8. If a tensor $S : BM^2 \longrightarrow BM$ satisfies the identities (17) in the domain of a chart x from the atlas of the manifold M_n, that is, if it is a Killing 1/2-tensor with respect to the connection ∇, then it is also a Killing tensor of type 1/2 with respect to the connection $\tilde{\nabla} = \nabla + S$.

Example. Let $f : BM^2 \longrightarrow FM$ be any symmetric field of 1/1 harmonic tensors on M_n. By definition we have (cf.[3]).

$$\nabla_{[i}f_{j]|l} = 0 \ \wedge \ g^{ij}\nabla_i f_{jl} = 0.$$

Let us put

$$S_{ij}^{\ \ k} = \Gamma_{ij}^{\ \ k} - \tilde{\Gamma}_{ij}^{\ \ k} = \nabla_i f_{jl} g^{lk}.$$

It is easy to see that $S_{ij}^{\ \ k}$ satisfies the assumptions of Theorem 2.3.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

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