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CONCERNING NON-LINEAR EQUATIONS WITH RIGHT INVERTIBLE OPERATORS

In the present paper we shall give some results concerning mixed boundary value problems and initial value problems for quasi-linear equations and some non-linear equations with right invertible operators.

Let X be a linear space over an arbitrary field of scalars F. Let $L(X)$ denote the set of linear operators A mapping linear subsets $\mathcal{D}_A$ (called domain of A) of X into X and $L_0(X) = \{A \in L(X): \mathcal{D}_A = X\}$. Write: $Z_A = \ker A = \{x \in \mathcal{D}_A : Ax = 0\}$.

An operator $D \in L(X)$ is said to be right invertible (cf. [1]), if there is an operator $R \in L_0(X)$ such that $\mathcal{D}_D \subset RX$ and $DR = I$, where I denotes the identity operator. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. The set of all right inverses of an operator $D \in R(X)$ will be denoted by $\mathcal{R}_D$. The set $Z_D = \ker D$ is said to be the space of constants for a $D \in R(X)$.

An operator $F \in L(X)$ is said to be an initial operator for a $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$, if F is a projection onto the space of constants (i.e. if $F^2 = F$, $FX = Z_D$) such that $FR = 0$. By Theorem 2.2 of [1] an operator $F \in L(X)$ is initial for $D \in R(X)$, if and only if $F = I - RD$ on $\mathcal{D}_D$ for an $R \in \mathcal{R}_D$. Denote by $\mathcal{F}_D$ the set of all initial operators for $D \in R(X)$. By Theorem 2 of [2], if F is an initial operator for $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$, then

(1) $\mathcal{R}_D = \{R + FA : A \in L_0(X)\}$, $\mathcal{F}_D = \{F(I - AD) : A \in L_0(X)\}$.

T h e o r e m 1. Suppose that $F_0,\ldots,F_{M+N-1}$ are initial operators for $D\in R(X)$ corresponding to $R_0,\ldots,R_{M+N-1}\in\mathcal{R}_D$ respectively. Write:

(2) $$Q(D)=\sum_{k=0}^{N} Q_k D^k,$$

where $Q_0,\ldots,Q_{N-1}\in L(X)$, $Q_N = I$.

Let A be a non-linear mapping of X into itself. Suppose moreover that -1 is not eigenvalue of the operator $\hat{Q}$, where

(3) $$\hat{Q}=\sum_{k=0}^{N-1} Q_k R_k \ldots R_{N-1},$$

then the mixed boundary value problem

(4) $$Q(D)D^M x + A(x) = y,\quad y\in X,\qquad (M\geqslant 0),$$

(5) $$F_k D^k x = y_k,\quad y_k\in Z_D\quad (k=0,1,\ldots,M+N-1)$$

is reduced to the following equation

(6) $$x+\hat{A}(x)=\hat{y},$$

where

(7) $$\hat{A}=R_0\ldots R_{M+N-1}\,(I+\hat{Q})^{-1}A,$$

(8) $$\hat{y}=R_0\ldots R_{M+N-1}\,(I+\hat{Q})^{-1}\,y_{N+M}+y_0+\sum_{k=1}^{M+N-1} R_0\ldots R_{k-1}\,y_k,$$

(9) $$y_{M+N}=y-\sum_{m=0}^{N-1} Q_m\left[\sum_{k=m+1}^{N-1} R_m\ldots R_{k-1}\,y_k+y_m\right].$$

P r o o f. Write equation (4) in the form

(10) $$Q(D)D^M x=\tilde{y},$$

where $\tilde{y}=y-A(x)$.

Theorem 1.3 in [3] implies that the problem (10), (5) is well-posed[x)] and has a unique solution

$$x = R_o\ldots R_{M+N-1}(I+\hat{Q})^{-1}\left[\tilde{y} - \sum_{m=0}^{N-1} Q_m\left(\sum_{k=m+1}^{N-1} R_m\ldots R_{k-1}\, y_k + y_m\right)\right] +$$

$$+ y_o + \sum_{k=1}^{M+N-1} R_o\ldots R_{k-1}\, y_k = R_o\ldots R_{M+N-1}(I+\hat{Q})^{-1}\Big[y - A(X) +$$

$$- \sum_{m=0}^{N-1} Q_m\left(\sum_{k=m+1}^{N-1} R_m\ldots R_{k-1}\, y_k + y_m\right)\Big] + y_o + \sum_{k=1}^{M+N-1} R_o\ldots R_{k-1}\, y_k =$$

$$= -\hat{A}(x) + R_o\ldots R_{M+N-1}(I+\hat{Q})^{-1}\, y_{M+N} + y_o + \sum_{k=1}^{M+N-1} R_o\ldots R_{k-1}\, y_k = -\hat{A}(x) + \hat{y}$$

which was to be proved.

This theorem immediately implies the following

Corollary 1. Suppose that all assumptions of Theorem 1 are satisfied. If the transformation $I+\hat{A}$, where $\hat{A}$ is defined by formula (7), is invertible, then the problem (4), (5) is well-posed and its unique solution is of the form

$$x = (I+\hat{A})^{-1}\,\hat{y}, \tag{11}$$

where $\hat{y}$ is defined by Formulae (8), (9).

Corollary 2. Suppose that all assumptions of Theorem 1 are satisfied and that moreover the operators $Q_o,\ldots,Q_{N-1}$ commute with D. If the transformation $I+\hat{A}$, where $\hat{A}$ is defined by formula (7), is invertible, then the mixed boundary value problem for the equation

$$D^M Q(D)x + A(x) = y, \qquad y \in X \qquad (M \geqslant 0), \tag{12}$$

with conditions (5) is well-posed and its unique solution is of the form (11).

Putting $F_k=F$, $R_k=R$ for $k=0,1,\ldots,M+N-1$ we obtain

x) i.e. this problem has a unique solution for every $y \in X$, $y_o,\ldots,y_{M+N-1} \in Z_D$ (cf. [1]).

Corollary 3. Suppose that F is an initial operator for $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$. Define $Q(D)$ and A as in Theorem 1. If the transformation $I+\hat{A}$, where $\hat{A}=R^{M+N}(I+\hat{Q})^{-1}A$, is invertible, then the initial value problem

$$(4) \qquad Q(D)D^M x + A(x) = y, \quad y \in X \quad (M \geqslant 0),$$

$$(13) \qquad FD^k x = y_k, \quad y_k \in Z_D \quad (k=0,1,\dots,M+N-1)$$

is well-posed and its unique solution is of the form

$$(14)\ x = (I+\hat{A})^{-1}\left\{R^{M+N}(I+\hat{Q})^{-1}\left[y - \sum_{m=0}^{N-1} Q_m\left(\sum_{k=m+1}^{N-1} R^{k-m} y_k + y_m\right)\right] + \sum_{k=0}^{M+N-1} R^k y_k\right\}.$$

Corollary 4. Suppose that all assumptions of Corollary 3 are satisfied and that $Q_0,\dots,Q_{N-1}$ commute with D. If the transformation $I+\hat{A}$, where $\hat{A} = R^{M+N}(I+\hat{Q})^{-1}A$, is invertible, then the initial value problem (12), (13) is well-posed and its unique solution is of the form (14).

Corollary 5. Suppose that all assumptions of Theorem 1 are satisfied. If the transformation $I+\hat{A}$, where $\hat{A}$ is defined by formula (7), is invertible, then every solution of equation (4) is of the form

$$(15)\ x = (I+\hat{A})^{-1}\left\{R_0\dots R_{M+N-1}(I+\hat{Q})^{-1}\left[y - \sum_{m=0}^{N-1} Q_m\left(\sum_{k=m+1}^{N-1} R_m\dots R_{k-1}z_k + z_m\right)\right] + z_0 + \sum_{k=1}^{M+N-1} R_0\dots R_{k-1} z_k\right\},$$

where $z_0,\dots,z_{M+N-1} \in Z_D$ are arbitrary.

If -1 is an eigenvalue of the operator $\hat{Q}$ defined by formula (3), we can obtain conditions of solvability of problems under considerations in a similar way as in Theorem 3.2 of [1]. As regards the invertibility of the operator $I+\hat{A}$ and the solvability of equation (6) we have the same situation.

This means that we can assume various conditions of solvability of the equation (6) leading to different solutions of our problems.

T h e o r e m 2. Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. Let $\{H_x\}_{x \in X}$ be a family of mappings which depends in a non-linear way on $x \in X$ and such that for every fixed $x \in X$ the operator $H_x \in L(X)$ is invertible. Then every solution of the equation

$$(16) \qquad Dx = H_x y,$$

where $y \in X$, is a solution of the equation

$$(17) \qquad x - RH_x y = z,$$

where $z \in Z_D$ is arbitrary.

Conversely, every solution of equation (17) is a solution of equation (16).

P r o o f. Suppose that $x \in X$ is a solution of equation (17). By our assumptions $Dz=0$. Thus we find

$$Dx - H_x y = Dx - DRH_x y = D(x-RH_x y) = Dz = 0.$$

This means that x is a solution of equation (16).

Conversely, suppose that $x \in X$ satisfies equation (16). Then $x = RH_x y + z$, where $z \in Z_D$ is arbitrary. Hence x is a solution of equation (17).

Corollary 6. Suppose that all assumptions of Theorem 2 are satisfied and that F is an initial operator for D corresponding to R. Then every solution of the initial value problem for the equation (16) with the condition

$$(18) \qquad Fx = x_0, \quad x_0 \in Z_D$$

is a solution of the equation

$$(19) \qquad x - RH_x y = x_0.$$

Conversely, every solution of the equation (19) is a solution of the problem (16), (18).

Indeed, by our assumptions $FR=0$ and $Fx_0=x_0$. If x satisfies the equation (19), then, by Theorem 2, x is a solution of equation (16). Moreover, $Fx = F(RH_xy + x_0) = FRH_xy + Fx_0 = x_0$. Hence the condition (18) is also satisfied. Conversely, suppose that x is a solution of the problem (16)-(18). Since x satisfies equation (16), by Theorem 2, x satisfies equation (17) for a $z \in Z_D$.

Since $z \in Z_D$, we have $Fz = z$. Thus

$$z = Fz = F(x-RH_xy) = Fx - FRH_xy = Fx = x_0$$

which proves that x satisfies equation (19).

Observe that Theorem 2 and Corollary 6 generalize the method of solving of ordinary differential equation of the form $y' = f(x)\,g(y)$ for equations with right invertible operators. Another way of generalization is the following.

T h e o r e m 3. Suppose that

1) $D \in R(X)$ and $R_0, \ldots, R_{M+N-1} \in \mathcal{R}_D$;

2) The operator $Q(D)$ is defined by formula (2);

3) The number -1 is not an eigenvalue of the operator

$$\tilde{Q} = \sum_{k=0}^{N-1} Q_k R_{M+k} \ldots R_{M+N-1} \; ; \tag{20}$$

4) The family $\{H_x\}_{x \in X}$ satisfies the assumptions of Theorem 2.

Then every solution of the equation

$$Q(D)\, D^M x = H_x\, y, \qquad y \in X \quad (M \geqslant 0) \tag{21}$$

satisfies the equation

$$\begin{aligned} x - R_0 \ldots R_{M+N-1} (I+\tilde{Q})^{-1} H_x y &= \\ = R_0 \ldots R_{M+N-1} (I+\tilde{Q})^{-1} \sum_{m=0}^{N-1} Q_m \Big(\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}\, z_k + z_m \Big) &+ \\ + z_0 + \sum_{k=1}^{M+N-1} R_0 \ldots R_{k-1}\, z_k, & \end{aligned} \tag{22}$$

where $z_o,\dots,z_{M+N-1} \in Z_D$ are arbitrary. Conversely, every solution of equation (22) satisfies equation (21).

P r o o f. By our assumption the operator $I+\widetilde{Q}$ is invertible. Write

$$(23)\qquad \hat{D} = Q(D)D^M;\quad \hat{R} = R_o\dots R_{M+N-1}\,(I+\widetilde{Q})^{-1}.$$

We find

$$\hat{D}\hat{R} = Q(D)D^M R_o\dots R_{M+N-1}\,(I+\widetilde{Q})^{-1} =$$

$$= Q(D)\left[D^M R_o\dots R_{M-1}\right]R_M\dots R_{M+N-1}\,(I+\widetilde{Q})^{-1} =$$

$$= Q(D)R_M\dots R_{M+N-1}(I+\widetilde{Q})^{-1} = \sum_{k=0}^{N} Q_k\, D^k\, R_M\dots R_{M+N-1}(I+\widetilde{Q})^{-1} =$$

$$= \sum_{k=0}^{N} Q_k\, R_{M+k}\dots R_{M+N-1}(I+\widetilde{Q})^{-1} = (I+\widetilde{Q})(I+\widetilde{Q})^{-1} = I.$$

Thus $\hat{D} \in R(X)$ and $\hat{R} \in \mathcal{R}_{\hat{D}}$. Theorem 1.3 in [3] implies that

$$(24)\quad Z_{\hat{D}} = \Big\{z:\ z = R_o\dots R_{M+N-1}(I+\widetilde{Q})^{-1}\sum_{m=0}^{N-1} Q_m\Big(\sum_{k=M+m+1}^{N-1} R_m\dots R_{k-1}z_k + z_m\Big) + z_o + \sum_{k=1}^{M+N-1} R_o\dots R_{k-1}\, z_k,\ z_o,\dots,z_{M+N-1} \in Z_D\Big\}.$$

This formula and Theorem 2 together imply the conclusion of our theorem.

From Theorem 2 we obtain the following corollary.

Corollary 7. Suppose that all assumptions of Theorem 3 are satisfied and that, moreover, $R_o = \dots = R_{M+N-1} = R$. Then every solution of equation (21) satisfies the equation

$$(25)\qquad x - R^{M+N}\,(I+\widetilde{Q})^{-1}\,H_x y =$$

$$= R^{M+N}(I+\widetilde{Q})^{-1}\sum_{m=0}^{N-1} Q_m \sum_{k=m}^{N-1} R^{k-m}\, z_k + \sum_{k=0}^{M+N-1} R^k z_k,$$

where $z_0,\dots,z_{M+N-1} \in Z_D$ are arbitrary and

$$\tilde{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}.$$

Conversely, every solution of equation (25) satisfies equation (21').

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