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ON TOTALLY UMBILICAL SURFACES IMMERSED IN RIEMANNIAN CONFORMALLY RECURRENT AND CONFORMALLY SYMMETRIC SPACES

1. Introduction

An n-dimensional ($n > 2$) Riemannian space is said to be an R_n-space if its curvature tensor satisfies the condition

$$\nabla_l R_{kjih} = \Phi_l R_{kjih} \tag{1}$$

for some vector Φ_j, where the symbol ∇ denotes covariant differentiation with respect to the metric of the space. If Φ_j is assumed to be non-zero and the space is non-flat then the space is called of recurrent curvature, or briefly a recurrent space [8], [9]. In the case $\Phi_j = 0$, the R_n-space reduces to the well-known Cartan symmetric space.

According to Chaki and Gupta [3], an n-dimensional ($n > 3$) Riemannian space is said to be conformally symmetric, if its Weyl conformal tensor

$$C_{kjih} = R_{kjih} - \frac{1}{n-2}(G_{kh} R_{ji} - G_{ki} R_{jh} + G_{ji} R_{kh} - G_{jh} R_{ki}) + \frac{R}{(n-1)(n-2)}(G_{kh} G_{ji} - G_{ki} G_{jh}), \tag{2}$$

where G_{ji} denotes the components of the fundamental tensor of the space, satisfies

$$\nabla_l C_{kjih} = 0. \tag{3}$$

It follows easily from (2) and (3) that every conformally flat n-space ($n > 3$) as well as every Cartan symmetric n-space

$(n > 3)$ is necessarily conformally symmetric. The inverse implication fails in general [7]. Conformally symmetric spaces have been considered by many authors [1], [5], [7].

If the Weyl conformal tensor of a space is non-zero and satisfies a relation of the form

$$\nabla_l C_{kjih} = \phi_l C_{kjih},$$

where ϕ_j is a non-zero vector then the space is called conformally recurrent [2]. It is easy to verify that every recurrent n-space $(n > 3)$ is necessarily conformally recurrent with the same vector of recurrence.

By a CR_n-space we shall mean a Riemannian n-space $(n > 3)$ whose Weyl conformal tensor satisfies the equality

$$\nabla_l C_{kjih} = \phi_l C_{kjih} \tag{4}$$

for some vector ϕ_j; if $\phi_j \neq 0$ and $C_{kjih} \neq 0$, then the CR_n-space is conformally recurrent and if $\phi_j = 0$, then the CR_n-space is conformally symmetric.

Let V^m be an m-dimensional Riemannian space immersed in an n-dimensional Riemannian space V^n and let $u^i = u^i(w^a)$ be the parametric expression of the subspace V^m in V^n, where (u^i) are coordinates in V^n and (w^a) are coordinates in V^m. Let $B_a^i = \partial u^i / \partial w^a$. If G_{ji} is the fundamental tensor of the space V^n, then g_{ba} defined by $g_{ba} = B_b^j B_a^i G_{ji}$ is the first fundamental tensor of the subspace V^m. In the sequel the indices h, i, j, k, l take on values $1, \dots, n$ and the indices a, b, c, d, e take on values $1, \dots, m$ $(m < n)$.

Let N_x^i $(x = m+1, \dots, n)$ be pairwise orthogonal unit normals to V^m. Then we have the relations

$$G_{ji} N_x^j N_x^i = \varepsilon_x, \quad G_{ji} N_y^j N_x^i = 0 \ (y \neq x), \quad G_{ji} N_x^j B_a^i = 0, \tag{5}$$

where ε_x is the indicator of the vector N_x^i. It is easy to verify that

(6) $$B_b^j B_a^i g^{ba} = G^{ji} - \sum_x \varepsilon_x N_x^j N_x^i .$$

The Euler-Schouten curvature tensor $H_{ba}{}^i$ of the subspace V^m is defined by

(7) $$H_{ba}{}^i = \nabla_b B_a^i ,$$

where ∇_a denotes covariant differentiation with respect to the fundamental tensor g_{ba} of V^m [6].

If $H_{ba}{}^i$ satisfies the relation

(8) $$H_{ba}{}^i = g_{ba} H^i ,$$

where the vector H^i is given by

(9) $$H^i = \frac{1}{m} g^{ba} H_{ba}{}^i$$

and is called the mean curvature vector, then V^m is called a totally umbilical surface.

Totally umbilical surfaces have been studied by Miyazawa and Chuman [6]. They have proved there among others the following theorems:

A totally umbilical surface V^m immersed in a symmetric space is a conformally symmetric one.

A totally umbilical surface V^m immersed in a recurrent space is a conformally recurrent one, if the recurrence vector is not orthogonal to the V^m.

The present paper is concerned with some generalizations of the above mentioned theorems.

2. Preliminary

If for $H_{ba}{}^i$ defined by (7) we put

(10) $$H_{ba}{}^i = \sum_x \varepsilon_x H_{bax} N_x^i ,$$

then the second fundamental tensor H_{bax} for N_x^i is given by

(11) $$H_{bax} = H_{ba}{}^{i} N_{xi}.$$

The Gauss and Codazzi equations for V^m can be written in the form

(12) $$K_{dcba} = R_{kjih} B_d^k B_c^j B_b^i B_a^h + \sum_x \varepsilon_x (H_{dax} H_{cbx} - H_{dbx} H_{cax})$$

and

(13) $$R_{kjih} B_d^k N_x^j B_c^i B_b^h = \nabla_c H_{dbx} - \nabla_b H_{dcx} + \sum \varepsilon_y (L_{bxy} H_{dcy} - L_{cxy} H_{dby})$$

respectively [4], where we put

(14) $$L_{axy} = (\nabla_a N_x{}^i) N_{yi} \quad (= - L_{ayx})$$

and K_{dcba} is curvature tensor for V^m.

We assume that the subspace V^m immersed in the space V^n is totally umbilical. Substituting (8) into (11), we have

(15) $$H_{bax} = g_{ba} H^i N_{xi}.$$

Putting $H_x = H^i N_{xi}$, we can rewrite (15), (9) and (8) in the form

(16) $$H_{bax} = g_{ba} H_x,$$

(17) $$H^i = \sum_x \varepsilon_x H_x N_x^i, \qquad H_{ba}{}^{i} = \sum_x \varepsilon_x H_x N_x^i g_{ba},$$

respectively.

From (17) and (5) it follows that

$$H_i H^i = \sum_x \varepsilon_x H_x H_x.$$

Substituting (16) into (12), we obtain

$$K_{dcba} = R_{kjih} B_d^k B_c^j B_b^i B_a^h + H_i H^i (g_{da} g_{cb} - g_{db} g_{ca}). \tag{18}$$

After differentiating (16) covariantly with respect to w^a and substituting the resulting equation and (16) into (13) we have

$$R_{kjih} B_d^k N_x^j B_c^i B_b^h = g_{bd} \nabla_c H_x - g_{dc} \nabla_b H_x + \sum_y \varepsilon_y H_y (L_{bxy} g_{dc} - L_{cxy} g_{db}). \tag{19}$$

Differentiating (18) covariantly with respect to w^a and using (14) and (19), we obtain

$$\nabla_e K_{dcba} = (\nabla_l R_{kjih}) B_e^l B_d^k B_c^j B_b^i B_a^h + \bar{H}_{edcba}, \tag{20}$$

where we put

$$\bar{H}_{edcba} = \frac{1}{2} \Big[\nabla_a (H_i H^i)(g_{ed} g_{cb} - g_{ec} g_{bd}) + \nabla_b (H_i H^i)(g_{ec} g_{da} - g_{ed} g_{ca}) + \nabla_c (H_i H^i)(g_{da} g_{eb} - g_{db} g_{ea}) + \nabla_d (H_i H^i)(g_{ea} g_{cb} - g_{eb} g_{ca}) \Big] + \nabla_e (H_i H^i)(g_{da} g_{cb} - g_{db} g_{ca}).$$

3. Main results

L e m m a. If V^m is a totally umbilical surface immersed in V^n $(3 < m < n)$ and if a vector Φ_j on V^n satisfies (4), then so does the vector $\Phi_a = B_a^i \Phi_i$ on V^m.

P r o o f. From (4) and (2) it follows that

$$\nabla_l R_{kjih} = \Phi_l R_{kjih} + \frac{1}{n-2} \Big[G_{kh}(\nabla_l R_{ji} - \Phi_l R_{ji}) - G_{ki}(\nabla_l R_{jh} - \Phi_l R_{jh}) + G_{ji}(\nabla_l R_{kh} - \Phi_l R_{kh}) - G_{jh}(\nabla_l R_{ki} - \Phi_l R_{ki}) \Big] - \frac{1}{(n-1)(n-2)} (\nabla_l R - \Phi_l R)(G_{kh} G_{ji} - G_{ki} G_{jh}). \tag{21}$$

If we set $\Phi_a = B_a^{\cdot i}\Phi_i$, then from (18) we obtain

(22) $$\Phi_l \, R_{kjih} \, B_e^{\,l} B_d^{\,k} B_c^{\,j} B_b^{\,i} B_a^{\,h} =$$

$$= \Phi_e \, K_{dcba} - \Phi_e \, H_i H^i (g_{da} \, g_{cb} - g_{db} \, g_{ca}).$$

Substituting (21) into (20) and using (22) we have

(23) $$\nabla_e K_{dcba} = \Phi_e \, K_{dcba} - \Phi_e \, H_i H^i (g_{da} \, g_{cb} - g_{db} \, g_{ca}) +$$

$$+ \frac{1}{n-2}\Big[g_{da}(\nabla_l R_{ji} - \Phi_l R_{ji}) B_e^{\,l} B_c^{\,j} B_b^{\,i} - g_{db}(\nabla_l R_{jh} - \Phi_l R_{jh}) B_e^{\,l} B_c^{\,j} B_a^{\,h} +$$

$$+ g_{cb}(\nabla_l R_{kh} - \Phi_l R_{kh}) B_e^{\,l} B_d^{\,k} B_a^{\,h} - g_{ca}(\nabla_l R_{ki} - \Phi_l R_{ki}) B_e^{\,l} B_d^{\,k} B_b^{\,i}\Big] +$$

$$- \frac{1}{(n-1)(n-2)} (\nabla_l R - \Phi_l R) B_e^{\,l} (g_{da} \, g_{cb} - g_{db} \, g_{ca}) + \bar{H}_{edcba}.$$

Considering (22) we conclude from (20)

(24) $$\nabla_e K_{dcba} - \Phi_e K_{dcba} = (\nabla_l R_{kjih} - \Phi_l R_{kjih}) B_e^{\,l} B_d^{\,k} B_c^{\,j} B_b^{\,i} B_a^{\,h} +$$

$$- \Phi_e \, H_i H^i (g_{da} \, g_{cb} - g_{db} \, g_{ca}) + \bar{H}_{edcba}.$$

By contracting (24) with g^{cb} and making use of (6) we obtain

(25) $$\nabla_e K_{da} - \Phi_e K_{da} = (\nabla_l R_{kh} - \Phi_l R_{kh}) B_e^{\,l} B_d^{\,k} B_a^{\,h} +$$

$$- (\nabla_l R_{kjih} - \Phi_l R_{kjih}) B_e^{\,l} B_d^{\,k} B_a^{\,h} \sum_x \varepsilon_x \, N_x^{\,j} N_x^{\,i} +$$

$$- (m-1)\Phi_e \, H_i H^i \, g_{da} + \bar{H}_{edcba} \, g^{cb}.$$

We find $\nabla_l R_{kjih} - \Phi_l R_{kjih}$ from (21) and substitute the obtained expression into (25). This, by virtue of (5), gives

$$(26)\qquad \nabla_e K_{da} - \phi_e K_{da} = \frac{m-2}{n-2}(\nabla_l R_{kh} - \phi_l R_{kh})B_e^l B_d^k B_a^h +$$

$$- \frac{1}{n-2} g_{da} \sum_x \varepsilon_x(\nabla_l R_{ji} - \phi_l R_{ji})B_e^l N_x^j N_x^i +$$

$$+ \frac{n-m}{(n-1)(n-2)}(\nabla_l R - \phi_l R)B_e^l g_{da} +$$

$$- (m-1)\,\phi_e\, H_i H^i g_{da} + \bar{H}_{edcba}\, g^{cb}.$$

Contracting the last equation with g^{da} and considering (6) and the definition of $\bar{H}_{edcba}$ we find

$$\sum_x \varepsilon_x(\nabla_l R_{ji} - \phi_l R_{ji})B_e^l N_x^j N_x^i = \frac{1}{2}\left[\frac{m-2}{m-1} + \frac{m(n-m)}{(n-1)(m-1)}\right](\nabla_l R - \phi_l R)B_e^l +$$

$$- \frac{n-2}{2(m-1)}(\nabla_e K - \phi_e K) + \frac{1}{2}(n-2)(m+2)\nabla_e(H_i H^i) - \frac{1}{2}m(n-2)\phi_e\, H_i H^i.$$

Substituting this equation into (26) and considering the definition of $\bar{H}_{edcba}$ we obtain

$$(\nabla_l R_{kh} - \phi_l R_{kh})B_e^l B_d^k B_a^h = \frac{n-2}{m-2}(\nabla_e K_{da} - \phi_e K_{da}) +$$

$$- \frac{n-2}{2(m-1)(m-2)}(\nabla_e K - \phi_e K)g_{da} + \frac{1}{2(n-1)}(\nabla_l R - \phi_l R)B_e^l g_{da} +$$

$$- \frac{1}{2}(n-2)\left[\nabla_e(H_i H^i)g_{da} + \nabla_a(H_i H^i)g_{ed} + \nabla_d(H_i H^i)g_{ea} - \phi_e H_i H^i g_{da}\right].$$

If we now substitute the expression thus obtained into (23) we find

$$\nabla_e K_{dcba} = \Phi_e K_{dcba} + \frac{1}{m-2}\Big[(\nabla_e K_{cb} - \Phi_e K_{cb})g_{da} - (\nabla_e K_{ca} - \Phi_e K_{ca})g_{db} +$$

$$+ (\nabla_e K_{da} - \Phi_e K_{da})g_{cb} - (\nabla_e K_{db} - \Phi_e K_{db})g_{ca}\Big] +$$

$$- \frac{1}{(m-1)(m-2)}(\nabla_e K - \Phi_e K)(g_{da}\, g_{cb} - g_{db}\, g_{ca})$$

or $\nabla_e \bar{C}_{dcba} = \Phi_e \bar{C}_{dcba}$, where $\bar{C}_{dcba}$ denotes the conformal tensor for V^m, which completes the proof.

C o r o l l a r y. A totally umbilical m-dimensional surface in a CR_n-space $(3<m<n)$ is a CR_m-space.

Using the above Lemma we shall prove

T h e o r e m 1. A totally umbilical surface V^m immersed in a conformally symmetric space $V^n(3<m<n)$ is conformally symmetric.

P r o o f. The zero vector satisfies (4) for V^n, hence by the lemma so does the zero vector on V^m, as desired.

T h e o r e m 2. If V^m is a totally umbilical and non--conformally flat surface immersed in a conformally recurrent space $V^n(3<m<n)$ such that the recurrence vector is not orthogonal to V^m, then V^m is also conformally recurrent.

P r o o f. The assertion of the theorem follows from the lemma in the same manner as that of Theorem 1.

Similarly we can prove the following theorem.

T h e o r e m 3. A totally umbilical surface V^m immersed in a conformally recurrent space $V^n(3<m<n)$ is conformally symmetric, if the recurrence vector is orthogonal to V^m.

Analogous theorems may be stated for totally geodesic surfaces.

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