DEMONSTRATIO MATHEMATICA
Vol. VIII No 1 1975

Wanda Sznitko

INVESTIGATION OF REGULAR CONTINUITY FOR SOME SINGULAR SURFACE INTEGRAL

1. Introduction

Let S be a closed Lapunow surface bounding a region D^+ in the space E_3. We denote by D^- the complement of the set $D^+ \cup S$ to the whole space.

We introduce the following notation (see [1], p.168)

$$\mathcal{D}(X,Y,Z) = \begin{bmatrix} 0 & X & Y & Z \\ X & 0 & -Z & Y \\ Y & Z & 0 & -X \\ Z & -Y & X & 0 \end{bmatrix}, \quad \mathcal{D}^*(X,Y,Z) = \begin{bmatrix} 0 & X & Y & Z \\ X & 0 & Z & -Y \\ Y & -Z & 0 & X \\ Z & Y & -X & 0 \end{bmatrix},$$

and

$$M(A,Q) = -\mathcal{D}^*\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\xi}\right) \frac{1}{|A-Q|} \mathcal{D}(\alpha,\beta,\gamma),$$

where $|A-Q|$ denotes the Euclidean distance between the points $A(x,y,z) \in D^+ \cup D^-$ and $Q(\zeta,\eta,\xi) \in S$, and $N(\alpha,\beta,\gamma)$ is the versor normal to the surface S at the point Q directed outside the region D^+.

Let the elements $q_i(Q)$, i=1,2,3,4, of the column

$$q(Q) = \begin{bmatrix} q_1(Q) \\ q_2(Q) \\ q_3(Q) \\ q_4(Q) \end{bmatrix} \tag{1}$$

be defined and continuous on the surface S. The column

$$(2)\qquad F(A) = \frac{1}{4\pi}\int_S M(A,Q)q(Q)dS_Q \ ,$$

called integral of Cauchy type, is defined for every $A \in D^+ \cup D^-$. It is known (see [1] p.170) that the column (2) is holomorphic in $D^+ \cup D^-$; that is, it satisfies in this set an eliptic system of equations

$$(3)\qquad \mathcal{D}\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \xi}\right)F(A) = 0 \ .$$

Moreover, if the elements of column (1) satisfy Hölder's condition on the surface S, then for every $P \in S$ there exists an integral (singular in the sense of Cauchy's principal value)

$$(4)\qquad \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q$$

which can be expressed by the formula

$$(5)\quad \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q = \frac{1}{4\pi}\int_S M(P,Q)\left[q(Q) - q(P)\right]dS_Q + \frac{1}{2}q(P) \ .$$

We have also the following formulas

$$(6)\qquad \begin{aligned} F^+(P) &= \frac{1}{2}\,q(P) + \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q \\ F^-(P) &= -\frac{1}{2}\,q(P) + \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q \end{aligned}$$

which correspond to Plemelj's formulas. Here $F^+(P)$ and $F^-(P)$ are boundary values for the column F(A) defined by the formulas

$$F^+(P) = \lim_{\substack{A \to P \in S \\ (A \in D^+)}} F(A) \ , \qquad F^-(P) = \lim_{\substack{A \to P \in S \\ (A \in D^-)}} F(A) \ .$$

Concerning the singular integral (4) W.Żakowski has proved a theorem (see [4] p.43) which is an analogue of Privalov s theorem known from the theory of singular integrals (over a closed flot curves) in the sense of Cauchy's principal value.

The aim of the present work is a generalization of the above mentioned theorem of Żakowski to the case of integral with parameter of the form (9).

2. The investigation of a singular integral

Let $g(Q,B) = [g_1(Q,B),\ g_2(Q,B),\ g_3(Q,B),\ g_4(Q,B)]$ denotes a column with elements defined for $Q \in S$, $B \in \Sigma$, where S and Σ are bounded closed Lapunow surfaces arbitrarily situated in the space E_3.

T h e o r e m. If the elements $g_i(Q,B), i=1,2,3,4$, of the column $g(Q,B)$ satisfy for every system of points $Q \in S$, $Q_1 \in S$ and $B \in \Sigma$, $B_1 \in \Sigma$ the conditions

$$(7) \qquad |g_i(Q,B)| \leqslant K ,$$

$$(8) \qquad |g_i(Q,B)-g_i(Q_1,B_1)| \leqslant K\left[|Q-Q_1|^h+|B-B_1|^{\mu}\right],\quad 0<h<1, 0<\mu\leqslant 1,$$

where K is a positive constant, then the elements $\phi_i(P,B)$, $i=1,2,3,4$, of the column $\phi(P,B)$ defined by a generalized singular integral of the Cauchy type

$$(9) \qquad \phi(P,B) = \frac{1}{4\pi}\int_S M(P,Q)g(Q,B)dS_Q ,$$

$(P \in S, B \in \Sigma)$ satisfy the inequalities

$$(10) \qquad |\phi_i(P,B)| \leqslant K(\tfrac{1}{2} + C)$$

$$(11) \qquad |\phi_i(P,B)| - \phi_i(P_1,B_1)| \leqslant C_1 K\left[|P-P_1|^h + |B - B_1|^{\mu-\varepsilon}\right],$$

where $P_1 \in S$, and C as well as C_1 are some positive constants independent of the column $g(Q,B)$, and ε is an arbitrary positive constant satisfying the condition $\mu > \varepsilon > \max(0, \mu - h)$.

P r o o f. We represent the integral (9) in the following form

$$\phi(P,B) = \frac{1}{4\pi}\int_S M(P,Q)\Big[g(Q,B) - g(P,B)\Big]\,dS_Q + \frac{1}{2}\,g(P,B)\ .$$

Next, taking into account inequalities (7),(8) and the following estimation of the elements $M_{ik}(P,Q)$ $(i,k=1,2,3,4)$ of the matrix $M(P,Q)$

$$(12)\qquad \left|M_{ik}(P,Q)\right| \leqslant \frac{1}{|P-Q|^2}$$

see ((4) p.39), we obtain

$$\left|\phi_i(P,B)\right| \leqslant \frac{1}{4\pi}\int_S \frac{4K|P-Q|^h}{|P-Q|^2}\,dS_Q + \frac{1}{2}\left|g_i(P,B)\right|.$$

This implies

$$(13)\qquad \left|\phi_i(P,B)\right| \leqslant K\left(\frac{1}{2} + C\right),$$

where

$$C = \sup_S \frac{1}{\pi}\int_S \frac{dS_Q}{|P-Q|^{2-h}}\ .$$

Hence inequality (10) has been proved.

To prove Hölder's condition (11) we consider the difference

$$(14)\quad \left|\phi_i(P,B) - \phi_i(P_1,B_1)\right| \leqslant \left|\phi_i(P,B) - \phi_i(P_1,B)\right| + \left|\phi_i(P_1,B) - \phi_i(P_1,B_1)\right|\ , \qquad i = 1,2,3,4.$$

The following inequality

$$(15)\qquad \left|\phi_i(P,B)-\phi_i(P_1,B)\right| \leqslant C'K\left|P-P_1\right|^h, \qquad i=1,2,3,4,$$

where C' is a positive constant independent of the column $g(Q,B)$, result immediately from Żakowski's theorem on preserving Hölder's class by the singular integral

$$\phi(P) = \frac{1}{4\pi}\int_S M(P,Q)g(Q)\,dS_Q\ .$$

Now we are going to prove Hölder's condition with respect to the parameter B. To this aim we consider the difference

$$\phi(P,B) - \phi(P,B_1) = \frac{1}{4\pi}\int_S M(P,Q)\left[g(Q,B) - g(P,B)\right]dS_Q + \frac{1}{2}g(P,B)+$$

$$- \frac{1}{4\pi}\int_S M(P,Q)\left[g(Q,B_1) - g(P,B_1)\right]dS_Q - \frac{1}{2}g(P,B_1) =$$

$$=\frac{1}{2}\left[g(P,B)-g(P,B_1)\right] +\frac{1}{4\pi}\int_S M(P,Q)\left\{\left[g(Q,B)-g(P,B) - g(Q,B_1)-g(P,B_1)\right]\right\}dS_Q .$$

Since in view of assumption (8) we have

$$(17)\qquad \frac{1}{2}\left|g_i(P,B)-g_i(P,B_1)\right|\leqslant \frac{1}{2}K\left|B-B_1\right|^{\mu}, \qquad i=1,2,3,4 ,$$

it remains to consider the integral

$$(18)\quad I = \frac{1}{4\pi}\int_S M(P,Q)\left\{\left[g(Q,B) - g(P,B)\right]-\left[g(Q,B_1) - g(P,B_1)\right]\right\} dS_Q .$$

To this integral we apply the classic method of potential theory (see,e.g. [3] p.593). It suffices to investigate the case,where $|B-B_1| < \delta/3$, and δ is a positive number so small that the measure of the angle φ between normals at the points P and Q of the part S_k of the surface S included inside the ball with radius δ and center at the point P satisfies the condition $\cos\varphi \geqslant 1/2$ (see [2] p.41).

Consider a circular cylinder W(P,R) with radius $R = |B-B_1|$ and axis along the normal to S at the point P. We assume that the height of the cylinder is selected in such a way that the set $W(P,R) \cap (S-S_k)$ is empty. Let σ_1 denote the part of the surface S cut off by the cylinder W(P,R). We have $\sigma_1 \subset S_k$. We place the origin of the coordinate system Pxyz at the point P of the surface S, where the axis Pz is directed according to the exterior normal at the point P with respect to the region D^+. Then the remaining axes lie in the plane π_P tangent to the surface S at the considered point P. Next we represent the integral I in the form of two integrals

$$(19) \qquad I = I^{\sigma_1} + I^{S-\sigma_1}$$

spread over the parts σ_1, and $S-\sigma_1$ of the surface S, respectively. In view of the relation (18), assumption (8) and the estimation (12) of the kernel M(P,Q) we have

$$(20) \qquad \left|I_i^{\sigma_1}\right| \leqslant \frac{1}{4\pi}\int_{\sigma_1}\frac{4}{|P-Q|^2}\left\{K|P-Q|^h + K|P-Q|^h\right\}dS_Q \leqslant \frac{2K}{\pi}\int_{\sigma_1}\frac{dS_Q}{|P-Q|^{2-h}}$$

for i=1,2,3,4. This implies

$$(21) \qquad \left|I_i^{\sigma_1}\right| \leqslant \frac{2K}{\pi}\int_{\sigma_1}\frac{dS_Q}{|P-Q|^{2-h}} \leqslant \frac{2K}{\pi}\int_{\Omega}\frac{dQ'}{|P-Q'|^{2-h}\cos\varphi},$$

where Ω is the projection of the surface σ_1 onto the plane π_p, φ denotes the measure of the angle between the axis Pz and the exterior normal to the surface S_k at the point $Q(\xi,\eta,\zeta)$ and $Q'(\xi,\eta)$ is the projection of the point $Q(\xi,\eta,\zeta)\in\sigma_1$ onto the plane π_p.

Introducing polar coordinates $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, where $\varrho=|P-Q'|$, in the plane π_p and making use of the inequality $\frac{1}{2}\leqslant\cos\varphi\leqslant 1$ which is satisfied at every point $Q\in S_k$, we obtain

$$(22) \qquad \left|I_i^{\sigma_1}\right| \leqslant \frac{2K}{\pi}\int_0^{2\pi}\int_0^{|B-B_1|}\frac{2\varrho}{\varrho^{2-h}}\,d\varrho\,d\theta \leqslant \frac{8K}{h}|B-B_1|^h = A_1|B-B_1|^h,$$

where $A_1>0$. We represent the integral $I^{S-\sigma_1}$ as follows

$$I^{S-\sigma_1} = \frac{1}{4\pi}\int_{S-\sigma_1}M(P,Q)\left\{\left[g(Q,B)-g(P,B)\right]+\left[g(P,B_1)-g(Q,B_1)\right]\right\}dS_Q =$$

$$(23) \qquad = \frac{1}{4\pi}\int_{S-\sigma_1}M(P,Q)\left\{\left[g(Q,B)-g(Q,B_1)\right]+\left[g(P,B_1)-g(P,B)\right]\right\}dS_Q .$$

Taking into account assumption (8) and conditions (12) we obtain

$$(24)\qquad \left|I_i^{S-\sigma_1}\right| \leqslant \frac{1}{4\pi}\int_{S-\sigma_1}\frac{4\cdot 2K|B-B_1|^{\mu}}{|P-Q|^2}\,dS_Q = 8K|B-B_1|^{\mu}\frac{1}{4\pi}\int_{S-\sigma_1}\frac{1}{|P-Q|^2}\,dS_Q\,,$$

$$i=1,2,3,4.$$

The last integral appearing in formula (24) can be split into a sum of two integrals:

$$(25)\qquad \frac{1}{4\pi}\int_{S-\sigma_1}\frac{1}{|P-Q|^2}\,dS_Q = \frac{1}{4\pi}\int_{S-\sigma_0}\frac{dS_Q}{|P-Q|^2} + \frac{1}{4\pi}\int_{\sigma_0-\sigma_1}\frac{dS_Q}{|P-Q|^2} = I^{S-\sigma_0} + I^{\sigma_0-\sigma_1},$$

where σ_0 is the part of the surface S cut off by the cylinder $W(P,\delta/3)$ with axis coinciding with the axis of the cylinder $W(P,|B-B_1|)$. For that part σ_0 of the surface S we hawe the inclusion $\sigma_1 \subset \sigma_0 \subset S_k$.

Projecting the part $\sigma_0 - \sigma_1$ of the surface S onto the plane π_P and using the inequality $\cos\varphi \geqslant 1/2$ we obtain

$$(26)\qquad I_i^{S-\sigma_0} \leqslant \frac{1}{4\pi}\left(\frac{3}{\delta}\right)^2 |S| = A_2\,,$$

and

$$(27)\qquad I_i^{\sigma_0-\sigma_1} \leqslant \frac{1}{4\pi}\int_0^{2\pi}\int_{|B-B_1|}^{\delta/3}\frac{2\varrho}{\varrho^2}\,d\varrho\, d\theta = \int_{|B-B_1|}^{\delta/3}\frac{1}{\varrho}\,d\varrho = \ln\frac{\delta}{3} - \ln|B-B_1| \leqslant$$

$$\leqslant \left|\ln\frac{\delta}{3}\right| + \left|\ln|B-B_1|\right| = A_3 + \left|\ln|B-B_1|\right|,$$

for $i=1,2,3,4$, where $|S|$ denotes the area of the surface S, $A_3 > 0$. By inequalities (24),(25),(26) and (27) we have

$$(28)\qquad \left|I_i^{S-\sigma_1}\right| \leqslant 8K|B-B_1|^{\mu}\left(A_2+A_3+\left|\ln|B-B_1|\right|\right) \leqslant$$

$$\leqslant 8K(A_2+A_3)\left|B-B_1\right|^{\mu} + 8K\left|B-B_1\right|^{\mu-\varepsilon}\left|B-B_1\right|^{\varepsilon}\left|\ln\left|B-B_1\right|\right|,$$

$$i = 1,2,3,4.$$

Since for sufficiently small $|B-B_1|$ the inequality $|B-B_1|^{\varepsilon}\ln|B-B_1| < 1$ holds, we can write condition (28) in the following form

$$\left|I_i^{S-\sigma_1}\right| \leqslant 8K(A_2+A_3+A_4+1)\left|B-B_1\right|^{\mu-\varepsilon}, \quad i=1,2,3,4, \tag{29}$$

where ε is a positive constant amaller than μ, and $A_4 = \sup\limits_{\varepsilon}\sup\limits_{B,B_1}|B-B_1|^{\varepsilon}$. Combining inequalities (22) and (29) we obtain

$$|I_i| \leqslant A_1K\left|B-B_1\right|^{h}+A_4K\left|B-B_1\right|^{\mu-\varepsilon} \leqslant A_5K\left|B-B_1\right|^{\mu-\varepsilon}, \tag{30}$$

$$i=1,2,3,4,$$

where $\varepsilon > \max(0, \mu-h)$, $A_5 > 0$. From (16), (17) and (30) it follows that

$$\left|\phi_i(P,B)-\phi_i(P,B_1)\right| \leqslant C''K\left|B-B_1\right|^{\mu-\varepsilon}, \quad i=1,2,3,4, \tag{31}$$

where the positive constant C'' depends on the choice of ε and on the surface S.

Finally, taking into account estimations (14), (15) and (31) we obtain inequality (11), where $C_1 = \max(C', C'')$. Hence inequality (11) has been proved in the case $|B-B_1| < \delta/3$. For $|B-B_1| \geqslant \frac{\delta}{3}$ we have

$$\left|\phi_i(P,B)-\phi_i(P_1,B_1)\right| \leqslant \left|\phi_i(P,B)\right| + \left|\phi_i(P_1,B_1)\right| \leqslant K(1+2C), \tag{32}$$

$$i=1,2,3,4.$$

Multiplying both sides of inequality (32) by $|B-B_1|$ we obtain

$$\frac{\delta}{3}\left|\phi_i(P,B)-\phi_i(P_1,B_1)\right| \leqslant \left|B-B_1\right|\left|\phi_i(P,B)-\phi_i(P_1,B_1)\right| \leqslant K(1+2C)\left|B-B_1\right|^{\lambda},$$

$$i=1,2,3,4,$$

$$\left|\phi_i(P,B)-\phi_i(P_1,B_1)\right| \leqslant \frac{3b}{\delta}(1+2C)\,K\left|B-B_1\right|^{\mu-\varepsilon}, \qquad i=1,2,3,4,$$

where $b = \sup\limits_{\Sigma} \left|B-B_1\right|^{1-\mu+\varepsilon}$. This ends the proof of the theorem.

The theorem proved above is true also in the special case, important for applications, where P=B and Σ=S; that is, it is true for the integral

$$\phi(P,P) = \frac{1}{4\pi}\int\limits_S M(P,Q)g(Q,P)dS_Q, \qquad P\in S, \qquad Q\in S,$$

where the elements of the column $g(P,Q)$ satisfy the inequality

$$\left|g_i(P,Q)-g_i(P_1,Q_1)\right| \leqslant K\left[\left|Q-Q_1\right|^h+\left|P-P_1\right|^{\mu}\right], o<h<1, \quad o<\mu\leqslant 1.$$

To prove this special case, it suffices to represent the integrals $\phi(P,P)$ and $\phi(P_1,P_1)$ in the following form

$$\phi(P,P) = \frac{1}{4\pi}\int\limits_S M(P,Q)\left[g(Q,P) - g(P,P)\right]dS_Q + \frac{1}{2}g(P,P),$$

$$\phi(P_1,P_1) = \frac{1}{4\pi}\int\limits_S M(P_1,Q)\left[g(Q,P_1) - g(P_1,P_1)\right]dS_Q + \frac{1}{2}g(P_1,P_1)$$

and then make direct use of the theorem of Żakowski ([4] p.37).

BIBLIOGRAPHY

[1] А.В. Б и ц а д з е (A.W. Bicadze): Краевые задачи для эллиптических уравнений второго порядка. Москва 1966.

[2] W. P o g o r z e l s k i: Równania całkowe i ich zastosowania, (Integral equations and their applications), t.II. Warszawa 1958.

[3] В.И. С м и р н о в (W.I. Smirnow): Курс высшей математики. Т.IV. Москва 1957.

[4] W. Ż a k o w s k i: On some properties of the Cauchy type integral in the Euclidean space E_3, Demonstratio Math. 1(1969)175-185.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

Received December 11, 1973.