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A NOTE ON THE MAXIMUM MODULUS OF $\frac{1}{\Gamma(z)}$

The purpose of this note is to give a very simple proof of the following theorem.

T h e o r e m. Let $f(z) = \frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$. Then $\log M(r,f) \sim r \log r$ (as $r \to \infty$)*).

P r o o f. We first bound $f(z)$ for $\operatorname{Re} z \geq 0$. Set $z = re^{i\theta}$, where $\cos\theta \geq 0$. Then since $(1+u)\, e^{-u} \leq 1$ for $u \geq 0$, we have

$$|f(z)|^2 = r^2\, e^{2\gamma r\cos\theta} \prod_{n=1}^{\infty} \left(1 + \frac{2r\cos\theta}{n} + \frac{r^2}{n^2}\right) e^{-\frac{2}{n} r\cos\theta} \leq$$

$$\leq r^2\, e^{2\gamma r\cos\theta} \prod_{n=1}^{\infty} \left(1 + \frac{r^2}{n^2}\right) \prod_{n=1}^{\infty} \left(1 + \frac{2r\cos\theta}{n}\right) e^{-\frac{2}{n} r\cos\theta} \leq$$

$$\leq r^2\, e^{2\gamma r\cos\theta} \prod_{n=1}^{\infty} \left(1 + \frac{r^2}{n^2}\right) = r^2\, e^{2\gamma r\cos\theta}\, \frac{e^{\pi r} - e^{-\pi r}}{2\pi r} \leq r^2\, e^{2\gamma r} e^{\pi r}.$$

Hence for $\operatorname{Re} z \geq 0$,

$$|f(z)| \leq re^{\left(\gamma + \frac{\pi}{2}\right) r}. \tag{1}$$

*) $M(r,f) = \max_{|z| \leq r} |f(z)|$.

We now bound $f(z)$ for $\operatorname{Re} z < 0$. Let n be a positive integer satisfying $\operatorname{Re}(z+n) \geqslant 0$. Using the relation $\Gamma(z+1) = z\Gamma(z)$, we have

$$\frac{1}{\Gamma(z)} = \frac{z(z+1)\ \ldots\ (z+n-1)}{\Gamma(z+n)} .$$

Hence by (1) we have

$$(2) \qquad \begin{aligned} |f(z)| &\leqslant r(r+1)\ \ldots\ (r+n-1)\,|f(z+n)| \leqslant \\ &\leqslant r(r+1)\ \ldots\ (r+n-1)(r+n)e^{(\gamma+\frac{\pi}{2})(r+n)} . \end{aligned}$$

Using (1) and (2) we, immediately, obtain

$$\begin{aligned} M(n,f) \leqslant n(n+1)\ \ldots\ (n+n)e^{(\gamma+\frac{\pi}{2})2n} &\leqslant 2^{2n+1} n^{n+1} e^{-n}\ e^{(\gamma+\frac{\pi}{2})2n} = \\ &= 2^{2n+1} n^{n+1} e^{(2\gamma+\pi-1)n} . \end{aligned}$$

Suppose $n \leqslant r \leqslant n+1$. Then

$$\begin{aligned} M(r,f) \leqslant M(n+1,f) &\leqslant 2^{2n+3}(n+1)^{n+2} e^{(2\gamma+\pi-1)(n+1)} \leqslant \\ &\leqslant 2^{2r+3}\ (r+1)^{r+2} e^{(2\gamma+\pi-1)(r+1)} . \end{aligned}$$

Hence, we have

$$(3) \qquad \limsup_{r\to\infty} \frac{\log M(r,f)}{r \log r} \leqslant 1 .$$

Now we obtain a lower bound for $M(r,f)$ by using

$$\frac{1}{\Gamma(-z)} = \frac{-z\,\Gamma(z)\ \sin \pi z}{\pi} .$$

Set $z = n + \frac{1}{2}$ and let $n \geqslant 2$. Then

$$M(n+\tfrac{1}{2},f) \geqslant \frac{(n+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\pi} \geqslant \frac{n\Gamma(n)}{\pi} = \frac{n!}{\pi} \geqslant \frac{1}{\pi}\, n^n\, e^{-n} .$$

Hence if $n + \frac{1}{2} \leqslant r \leqslant n + \frac{3}{2}$,

$$M(r,f) \geqslant M(n + \tfrac{1}{2}, f) \geqslant \frac{1}{\pi} n^n e^{-n} \geqslant \frac{1}{\pi} (r - \tfrac{3}{2})^{(r-\frac{3}{2})} \quad e^{-(r-\frac{1}{2})} \quad .$$

Hence, we have

$$\text{(4)} \qquad \liminf_{r\to\infty} \frac{\log M(r,f)}{r \log r} \geqslant 1.$$

The result follows from (3) and (4).

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