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ON GENERALIZATIONS OF PRODUCT-CONFORMAL KILLING VECTORS

The idea of product-conformal Killing vector field in a locally product Riemannian space was originally introduced by Tachibana [1]. Subsequently, Yamaguchi generalized such vector fields in [2]. In the present paper, p-product-conformal Killing vector fields (which, when p=1, reduce to product-conformal Killing vector fields as given in [2]) have been defined and corresponding properties are obtained.

1. Introduction

The prerequisites for this paper are to be found in [3], [4] and [6]. However, we list here some notations and definitions, which have been frequently used.

Locally product Riemannian space M is an n-dimensional space with a mixed tensor $F_i^{\ h}$ and with a positive definite Riemannian metric g_{ji}, satisfying the following conditions

$$F_j^{\ i} F_i^{\ h} = A_j^h \tag{1.1}$$

and

$$F_j^{\ t} F_i^{\ s} g_{ts} = g_{ji} \, . \tag{1.2}$$

M is said to be locally decomposable Riemannian space if the following condition is also satisfied

$$F_i^{\ h}{}_{,j} = 0 \quad \text{or} \quad F_{ih,j} = 0, \tag{1.3}$$

where comma (,) followed by an index denotes the covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$.

For brevity we shall denote $X_{i_1 i_2 \cdots i_p}$ by X_{i_1/i_p} . From (1.3), it is easy to verify that the curvature tensor $R_{kji}{}^{h}$ is pure in all its indices.

D e f i n i t i o n 1.1. In a compact orientable locally decomposable Riemannian space M, a vector field X_i is said to be p-decomposable, if $X_{i,i_2/i_p j}$ is pure in i and j [6].

G r e e n's T h e o r e m. In a compact orientable Riemannian space M, we have

$$\int_M X^i{}_{,i}\, d\sigma = 0$$

for an arbitrary vector field X^i [4].

L e m m a 1.1. In a compact orientable locally decomposable Riemannian space, a necessary and sufficient condition for a vector field X_i to be p-decomposable is [6]

$$g^{st} X_{i,i_2/i_p st} - F^{jt}(X_{s,i_2/i_p jt} F_i{}^{s}) = 0.$$

We shall also assume that the space is of class C^{p+2} and all other tensors considered in this paper are of class C^{p+1}.

2. p-product-conformal Killing vector fields

D e f i n i t i o n 2.1. A vector field X_i is said to be p-product-conformal Killing or for brevity, p-PC-Killing, if there exist tensor fields ρ_{i_2/i_p} and σ_{i_2/i_p} such that

$$(2.1)\qquad X_{i,i_2/i_p j} + X_{j,i_2/i_p i} = 2(\rho_{i_2/i_p} g_{ji} + \sigma_{i_2/i_p} F_{ji}).$$

Transvecting equation (2.1) with g^{ji} and F^{ji} respectively, we get

$$(2.2)\qquad X^r{}_{,i_2/i_p r} = n\rho_{i_2/i_p} + F\sigma_{i_2/i_p}$$

and

$$(2.3)\qquad X^{*r}{}_{,i_2/i_p r} = F\rho_{i_2/i_p} + n\sigma_{i_2/i_p} ,$$

where

$$F \equiv F_a{}^a = g^{ji} F_{ji}$$

and

$$X^{*r} = X_j F^{jr} .$$

We call a p-PC-Killing vector field X_i to be <u>special p-PC-Killing</u>, if the tensor fields ϱ_{i_2/i_p} and σ_{i_2/i_p} satisfy the relation

$$\varrho_{i_2/i_p,i} = F_i{}^k \sigma_{i_2/i_p,k} . \tag{2.4}$$

Solving equations (2.2) and (2.3), we get

$$\varrho_{i_2/i_p} = \left(nX^r{}_{,i_2/i_p r} - FX^{*r}{}_{,i_2/i_p r}\right) / (n^2 - F^2) \tag{2.5}$$

and

$$\sigma_{i_2/i_p} = \left(nX^{*r}{}_{,i_2/i_p r} - FX^r{}_{,i_2/i_p r}\right) / (n^2 - F^2) . \tag{2.6}$$

Now, differentiating covariantly equation (2.1) with respect to k and multiplying by g^{jk}, we obtain

$$g^{jk} X_{i,i_2/i_p jk} + g^{jk} X_{j,i_2/i_p ik} - 2\varrho_{i_2/i_p,i} - 2F_i{}^k \sigma_{i_2/i_p,k} = 0$$

or

$$g^{jk} X_{i,i_2/i_p jk} + \left(X^k{}_{,i_2/i_p ik} - X^k{}_{,i_2/i_p ki}\right) + X^k{}_{,i_2/i_p ki} -$$
$$- 2\varrho_{i_2/i_p,i} - 2F_i{}^k \sigma_{i_2/i_p,k} = 0.$$

Using Ricci identity and equations (2.5) and (2.6), we have

$$g^{jk} X_{i,i_2/i_p jk} + R_{ik} X^k{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{kii_t}{}^a X^k{}_{,i_2/i_{t-1} a i_{t+1}/i_p} + \tag{2.7}$$
$$+ \frac{1}{n^2 - F^2}\Big\{(n^2 - F^2 - 2n) X^k{}_{,i_2/i_p ki} + 2FX^{*k}{}_{,i_2/i_p ki} -$$

$$(2.7) \qquad - 2nF_i^{\,j} X^{*k}{}_{,i_2/i_pkj} - 2FF_i^{\,j} X^k{}_{,i_2/i_pkj}\} = 0 ,$$

which is thus a necesarry condition for a vector field to be p-PC-Killing.

Now, we shall prove that if a p-PC-Killing vector field is p-decomposable [6] then it is special p-PC-Killing, that is,

$$\varrho_{i_2/i_p,k} = F_k^{\,i} \sigma_{i_2/i_p,i}$$

or

$$(2.8) \qquad (n^2 - F^2)\varrho_{i_2/i_p,k} = \left[F_k^{\,i}(nF_j^{\,r} X^j{}_{,i_2/i_pri} - FX^r{}_{,i_2/i_pri})\right]$$

It is easy to verify that

$$F_k^{\,i}(nF_j^{\,r} X^j{}_{,i_2/i_pri} - FX^r{}_{,i_2/i_pri}) = nF_k^{\,i} F_j^{\,r}(X^j{}_{,i_2/i_pir}) +$$

$$+ nF_k^{\,i} F_j^{\,r}(X^j{}_{,i_2/i_pri} - X^j{}_{,i_2/i_pir}) - FF_k^{\,i}(X^r{}_{,i_2/i_pir}) -$$

$$- FF_k^{\,i}(X^r{}_{,i_2/i_pri} - X^r{}_{,i_2/i_pir}) = nF_j^{\,r}(X^j{}_{,i_2/i_pi} F_k^{\,i})_{,r} +$$

$$+ nF_k^{\,i} F_j^{\,r}\Big(R_{ira}{}^j X^a{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{iri_t}{}^a X^j{}_{,i_2/i_{t-1}ai_{t+1}/i_p}\Big) -$$

$$- F(X^r{}_{,i_2/i_pi} F_k^{\,i})_{,r} - FF_k^{\,i}\Big(R_{ira}{}^r X^a{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{iri_t}{}^a X^r{}_{,i_2/i_{t-1}ai_{t+1}/i_p}\Big) .$$

Since the curvature tensor is pure in all its indices and vector field X_i is p-decomposable, therefore

$$F_k^{\,i}\, nF_j^{\,r} X^j{}_{,i_2/i_pri} - FX^r{}_{,i_2/i_pri} =$$

$$= nF_j^{\,r} F_t^{\,j}(X^t{}_{,i_2/i_pkr}) + n\Big(R_{kja}{}^j X^a{}_{,i_2/i_p} -$$

$$- \sum_{t=2}^{p} R_{kji_t}{}^a X^j{}_{,i_2/i_{t-1}ai_{t+1}/i_p}\Big) - FF_j^{\,r}(X^j{}_{,i_2/i_pkr}) +$$

$$+ FF_k^{\ i}\Big(R_{ia}\, X^a{}_{,i_2/i_p} + \sum_{t=2}^{p} R_{iri_t}{}^{a} X^r{}_{,i_2/i_{t-1}\, a\, i_{t+1}/i_p}\Big) =$$

$$= \Big\{ n\big(X^j{}_{,i_2/i_p jk}\big) - FF_j^{\ r}\big(X^j{}_{,i_2/i_p rk}\big)\Big\} + FF_j^{\ r}\big(X^j{}_{,i_2/i_p rk} -$$

$$- X^j{}_{,i_2/i_p kr}\big) + FF_k^{\ i}\Big(R_{ia}X^a{}_{,i_2/i_p} + \sum_{t=2}^{p} R_{iri_t}{}^{a}\, X^r{}_{,i_2/i_{t-1} a i_{t+1}/i_p}\Big) =$$

$$= \big(n^2-F^2\big)\varrho_{i_2/i_p,k} + F_j^{\ r}\Big(R_{kra}{}^{j}X^a{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{kri_t}{}^{a}X^j{}_{,i_2/i_{t-1}\, a\, i_{t+1}/i_p}\Big) +$$

$$+ FF_k^{\ i}\Big(R_{ia}X^a{}_{,i_2/i_p} + \sum_{t=2}^{p} R_{iri_t}{}^{a}\, X^r{}_{,i_2/i_{t-1} a\, i_{t+1}/i_p}\Big) = (n^2-F^2)\varrho_{i_2/i_p,k}$$

by virtue of purity of the curvature tensor. Hence we get

$$\varrho_{i_2/i_p,k} = F_k^{\ i}\, \sigma_{i_2/i_p,i}\ ,$$

which provides the proof of the following

T h e o r e m 2.1. In a locally product Riemannian space M, if a p-PC-Killing vector field is p-decomposable then it is special p-PC-Killing.

Next, multiplying equation (2.1) by g^{ih}, g^{jh} and g^{kh} and differentiating covariantly with respect to i,j and k respectively, we get

$$(2.9)\quad X^h{}_{,i_2/i_p ji} + X_{j,i_2/i_p i}{}^{h} = 2\big(\varrho_{i_2/i_p,i}\, A_j^h + \sigma_{i_2/i_p,i}\, F_j^{\ h}\big)$$

$$(2.10)\quad X_{i,i_2/i_p j}{}^{h} + X^h{}_{,i_2/i_p ij} = 2\big(\varrho_{i_2/i_p,j}\, A_i^h + \sigma_{i_2/i_p,j}\, F_i^{\ h}\big)$$

and

$$(2.11)\quad X_{i,i_2/i_p j}{}^{h} + X_{j,i_2/i_p i}{}^{h} = 2\big(\varrho_{i_2/i_p,}{}^{h}\, g_{ji} + \sigma_{i_2/i_p,}{}^{h}\, F_{ji}\big).$$

Adding equations (2.9) and (2.10) and subtracting (2.11),we obtain

$$T^h_{i_2/i_pji} + T^h_{i_2/i_pij} =$$

(2.12)
$$= 2\left(\varrho_{i_2/i_p,i}\, A^h_j + \varrho_{i_2/i_p,j}\, A^h_i - \varrho_{i_2/i_p,}{}^h\, g_{ji} + \right.$$

$$\left. + \sigma_{i_2/i_p,i}\, F_j{}^h + \sigma_{i_2/i_p,j}\, F_i{}^h - \sigma_{i_2/i_p,}{}^h F_{ji}\right),$$

where

$$T^h_{i_2/i_pji} \equiv X^h{}_{,i_2/i_pji} + X_{j,i_2/i_pi}{}^h - X'_{j,i_2/i_pi}{}^h =$$

$$= X^h{}_{,i_2/i_pji} + R_{aji}{}^h X^a{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{i_t}{}^a{}_i{}^h X_{j,i_2/i_{t-1}\,a\,i_{t+1}/i_p}\,.$$

From equation (2.12), it is easy to verify that

(2.13)
$$g^{ji}\, T^h_{i_2/i_pji} = -(n-2)\, \varrho_{i_2/i_p,}{}^h + 2\sigma_{i_2/i_p,i}\, F_i{}^h - F\sigma_{i_2/i_p,}{}^h$$

and

(2.14)
$$F_t{}^h F^{ji}\, T^t_{i_2/i_pji} = 2\varrho_{i_2/i_p,}{}^h - FF_t{}^h \varrho_{i_2/i_p,}{}^t +$$

$$+ 2\sigma_{i_2/i_p,}{}^i\, F_i{}^h - nF_t{}^h \sigma_{i_2/i_p,}{}^t$$

Subtracting equation (2.14) from equation (2.13), we get

$$g^{ji}\, T^h_{i_2/i_pji} - F_t{}^h F^{ji}\, T^t_{i_2/i_pji} =$$

$$= -n\varrho_{i_2/i_p,}{}^h + FF_t{}^h \varrho_{i_2/i_p,}{}^t - F\sigma_{i_2/i_p,}{}^h + nF_t{}^h \sigma_{i_2/i_p,}{}^t$$

or

(2.15)
$$g^{ji}\, X^h{}_{,i_2/i_pji} - F^{ji}\left(X^t{}_{,i_2/i_pji} F_t{}^h\right) =$$

$$= -n\,\varrho_{i_2/i_p,}{}^h + FF_t{}^h \varrho_{i_2/i_p,}{}^t - F\sigma_{i_2/i_p,}{}^h + nF_t{}^h \sigma_{i_2/i_p,}{}^t.$$

Assuming that the vector field X_i is special p-PC-Killing, we get

$$g^{ji} X^h{}_{,i_2/i_pji} - F^{ji}(X^t{}_{,i_2/i_pji} F_t{}^h) =$$

$$= - n\varrho_{i_2/i_p,}{}^h + F\sigma_{i_2/i_p,}{}^h - F\sigma_{i_2/i_p,}{}^h + n\varrho_{i_2/i_p,}{}^h$$

or

$$(2.16) \qquad g^{ji} X^h{}_{,i_2/i_pji} - F^{ji}(X^t{}_{,i_2/i_pji} F_t{}^h) = 0 ,$$

which is a necessary and sufficient condition for a vector field to be p-decomposable in a compact orientable locally decomposable Riemannian space M [6]. Thus we have the following

T h e o r e m 2.2. In a compact orientable locally decomposable Riemannian space M, a special p-PC-Killing vector field is p-decomposable.

3. Integral formula

In this section we establish an integral formula for a compact orientable locally decomposable Riemannian space M and use it to obtain a necessary and sufficient condition for a p-PC-Killing vector field.

Now consider a vector field X_i in M and define tensor field S_{ji_2/i_pi} by

$$(3.1) \qquad S_{ji_2/i_pi} \equiv X_{i,i_2/i_pj} + X_{j,i_2/i_pi} - 2\varrho_{i_2/i_p} g_{ji} - 2\sigma_{i_2/i_p} F_{ji} ,$$

where ϱ_{i_2/i_p} and σ_{i_2/i_p} are given by (2.5) and (2.6) respectively. We notice that $S_{ji_2/i_pi} = 0$ is equivalent to the fact that the vector field X_i is a p-PC-Killing.

Putting

$$S^2 = S_{ji_2/i_pi} S^{ji_2/i_pi}$$

and calculating it by the aid of equation (3.1), we get

$$S^2 = \left(X_{i,i_2/i_pj} + X_{j,i_2/i_pi} - 2\rho_{i_2/i_p} g_{ji} - 2\sigma_{i_2/i_p} F_{ji}\right)\times$$
$$\times\left(X^{i,i_2/i_pj} + X^{j,i_2/i_pi} - 2\rho^{i_2/i_p} g^{ji} - 2\sigma^{i_2/i_p} F^{ji}\right),$$

where right hand member is the sum of the following five expressions A,B,C,D and E

$$A = 2(X_{i,i_2/i_pj})(X^{i,i_2/i_pj}), \qquad B = 2(X_{i,i_2/i_pj})(X^{j,i_2/i_pi}),$$
$$C = -8(\rho_{i_2/i_p} X^{i,i_2/i_p}{}_{i} + \sigma_{i_2/i_p} X^{*i,i_2/i_p}{}_{i}),$$
$$D = 4n(\rho_{i_2/i_p}\rho^{i_2/i_p} + \sigma_{i_2/i_p}\sigma^{i_2/i_p}), \qquad E = 8F\cdot\rho_{i_2/i_p}\sigma^{i_2/i_p}.$$

Using equations (2.5) and (2.6) in the above expressions, we get

$$\frac{1}{2}S^2 = \left(X_{i,i_2/i_pj}\right)\left(X^{i,i_2/i_pj}\right) + \left(X_{i,i_2/i_pj}\right)\left(X^{j,i_2/i_pi}\right) +$$
$$+\frac{1}{n^2-F^2}\left[4F\left(X^{i,i_2/i_p}{}_{i}\right)\left(X^{*j}{}_{,i_2/i_pj}\right) - 2n\left\{\left(X^{i,i_2/i_p}{}_{i}\right)\left(X^{j}{}_{,i_2/i_pj}\right) +\right.\right.$$
$$\left.\left. + \left(X^{*i}{}_{,i_2/i_pi}\right)\left(X^{*j,i_2/i_p}{}_{j}\right)\right\}\right]$$

or

$$(3.2) \qquad \frac{1}{2}S^2 = X^{i,i_2/i_pj}\, S_{ji_2/i_pi}\,.$$

Next, we shall calculate $S_{ji_2/i_pi,}{}^{j}$ which is sum of the following three expressions U,V and W

$$U = g^{jk} X_{i,i_2/i_pjk} + X^{j}{}_{,i_2/i_pij},$$
$$V = -2\rho_{i_2/i_p,i} = -\frac{2}{n^2-F^2}\left[nX^{k}{}_{,i_2/i_pki} - FX^{*k}{}_{,i_2/i_pki}\right],$$

$$W = -2F_i{}^j \sigma_{i_2/i_p,j} = -\frac{2}{n^2-F^2}\left[nF_i{}^j X^{*k}{}_{,i_2/i_p kj} - FF_i{}^j X^k{}_{,i_2/i_p kj}\right]$$

by virtue of equations (2.5) and (2.6). Thus

$$(3.3) \qquad g^{jk} S_{j i_2/i_p i,k} = g^{jk} X_{i,i_2/i_p jk} + R_i{}^k X_{k,i_2/i_p} -$$

$$- \sum_{t=2}^{p} R_{kii_t}{}^a X^k{}_{,i_2/i_{t-1} a i_{t+1}/i_p} + \frac{1}{n^2-F^2}\left\{\left(n^2-F^2-2n\right) X^k{}_{,i_2/i_p k i} +\right.$$

$$\left. + 2FX^{*k}{}_{,i_2/i_p k i} - 2nF_i{}^j X^{*k}{}_{,i_2/i_p kj} + 2nFF_i{}^j X^k{}_{,i_2/i_p kj}\right\} .$$

On the other hand

$$(3.4) \qquad \begin{aligned} & g^{jk}\left\{S_{j i_2/i_p i}\, X^{i,i_2/i_p}\right\}_{,k} = \\ & = g^{jk} S_{j i_2/i_p i,k}\, X^{i,i_2/i_p} + S_{j i_2/i_p i}\, X^{i,i_2/i_p j} . \end{aligned}$$

In view of (3.1) and (3.3), the above equation is equivalent to

$$(3.5) \qquad \begin{aligned} & g^{jk}\left\{S_{j i_2/i_p i}\, X^{i,i_2/i_p}\right\}_{,k} = \\ & = \left\{\text{R.H.S of equation (3.3)}\right\} X^{i,i_2/i_p} + \frac{1}{2} S^2 . \end{aligned}$$

Hence application of Green's theorem provides the proof of the following

T h e o r e m 3.1. In a compact orientable locally decomposable Riemannian space M, the following integral formula is valid for any vector field X_i

$$\int_M \left[\left\{g^{jk} X_{i,i_2/i_p jk} + R_{ik} X^k{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{kii_t}{}^a X^k{}_{,i_2/i_{t-1} a i_{t+1}/i_p} +\right.\right.$$

$$+ \frac{1}{n^2-F^2}\left((n^2-2n-F^2) X^k{}_{,i_2/i_p k i} + 2FX^{*k}{}_{,i_2/i_p k i} - 2nF_i{}^j X^{*k}{}_{,i_2/i_p kj} -\right.$$

$$+ 2FF_i{}^j X^k{}_{,i_2/i_p k_j} \Big) \Big\} \; X^{i,i_2/i_p} + \frac{1}{2} S^2 \Big] d\sigma = 0 ,$$

where $d\sigma$ is the volume element of M.

Theorem (3.1) yields

T h e o r e m 3.2. In a compact orientable locally decomposable Riemannian space M, a necessary and sufficient condition for a vector field X_i to be p-PC-Killing is that

(3.6) R.H.S. of equation (3.3) = 0 .

P a r t i c u l a r c a s e s

C a s e 1. If p = 1, then p-PC-Killing vector field becomes PC-Killing, for which a nacessary and suffieient condition is [2]

$$g^{jk} X_{i,jk} + R_{ik} X^k + \big[(n^2-2n-F^2) X^k{}_{,ki} + 2FX^{*k}{}_{,ki} -$$

$$- 2nF_i{}^j X^{*k}{}_{,kj} + 2FF_i{}^j X^k{}_{,kj} \big] / (n^2-F^2) = 0 .$$

C a s e 2. If $\sigma_{i_2/i_p} = 0$, then p-PC-Killing vector field reduces to p-conformal Killing vector field [5] and in this case equation (3.6) reduces to

$$g^{jk} X_{i,i_2/i_p jk} + R_{ik} X^k{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{kii_t}{}^a X^k{}_{,i_2/i_{t-1}\, a\, i_{t+1}/i_p} +$$

$$+ \frac{n-2}{n} X^k{}_{,i_2/i_p ki} = 0 .$$

C a s e 3. If p = 1 and $\sigma_{i_2/i_p} = 0$, then equation (3.6) changes to

$$g^{jk} X_{i,jk} + R_{ik} X^k + \frac{n-2}{n} X^k{}_{,ki} = 0,$$

which is a necessary and sufficient condition for a vector field to be conformal Killing [3].

C a s e 4. If a vector field X_i is a special p-PC-Killing, then we have

$$\varrho_{i_2/i_p,i} = F_i^{\ k}\sigma_{i_2/i_p,k}$$

or

$$nX^k{}_{,i_2/i_pki} - FX^{*k}{}_{,i_2/i_pki} = F_i^{\ k}\left(nX^{*j}{}_{,i_2/i_pjk} - FX^j{}_{,i_2/i_pjk}\right).$$

In this case theorem (3.1) provides the proof of the following

C o r o l l a r y 3.1. In a compact orientable locally decomposable Riemannian space M the integral formula

$$\int_M \Big[\Big\{g^{jk}X_{i,i_2/i_pjk} + R_{ik}X^k{}_{,i_2/i_p} - \sum_{t=2}^{p} R_{kii_t}{}^{a}X^k{}_{,i_2/i_{t-1}a\,i_{t+1}/i_p} + \frac{1}{n^2-F^2}\Big((n^2-4n-F^2)X^k{}_{,i_2/i_pki} + 4FX^{*k}{}_{,i_2/i_pki}\Big)\Big\}X^{i,i_2/i_p} + \frac{1}{2}S^2\Big]\,d\sigma = 0$$

is valid for any vector field X_i, where $d\sigma$ is the volume element of M.

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