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Edmund Pluciński

ON SOME QUESTIONS CONCERNING THE EQUATION OF KOLMOGOROV TYPE FOR NON-MARKOVIAN DISCONTINUOUS PROCESSES

Let $(\mathcal{X},B)$ be a measurable space, T a finite or infinite interval on the real axis and let $X(t)$, $t\in T$ be a stochastic process with a phase space $\mathcal{X}$. Let

$$P_{n+1}(t_0,x_0,t_1,x_1,\dots,t_n,x_n,t_{n+1},A) = P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A),$$

where $t_0,t_1,\dots,t_{n+1}\in T, t_0\leqslant t_1\leqslant\dots\leqslant t_n<t_{n+1}$, $x_0,x_1,\dots,x_n\in\mathcal{X}$ $A\in B$, $\bar{t}_n=[t_0,t_1,\dots,t_n]$, $\bar{x}_n=[x_0,x_1,\dots,x_n]$ are the transition probabilities in this process, i.e.

1^0. For fixed $\bar{t}_n,\bar{x}_n,t_{n+1}$ the function $P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A)$ is a probability measure on B.

2^0. For fixed $\bar{t}_n,t_{n+1},A$ the function $P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A)$ is measurable in $\bar{x}_n$ with respect to B.

3^0. $P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A)$ satisfies the equation

$$P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+2},A) =$$

$$=\int_{\mathcal{X}} P_{n+2}(\bar{t}_n,\bar{x}_n t_{n+1},z,t_{n+2},A)P_{n+1}(t_n,x_n,t_{n+1},dz).$$

We assume further that

$$P_{n+1}(\bar{t}_n,\bar{x}_n,t_n,A) = \mathcal{X}_A(x_n) \tag{2}$$

where χ_A is the characteristic function of the set A.

In paper [3] it was estabilished that under certain uniformity conditions the transition probabilities satisfy the equations

$$\frac{\partial P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A)}{\partial t_{n+1}} =$$

(3) $$= -\int_A q_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},z)P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},dz) +$$

$$+\int_{\mathcal{X}} q_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},z)\Pi_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},z,A)P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},dz)$$

$$\left.\frac{P_{n+2}(\bar{t}_n,\bar{x}_n,t_{n+1},x_n,t_{n+2},A)}{\partial t_{n+1}}\right|_{t_{n+1}=t_n} =$$

(4) $$= q_n(\bar{t}_n,\bar{x}_n)\Big[P_{n+2}(\bar{t}_n,\bar{x}_n,t_n,x_n,t_{n+2},A) -$$

$$- \lim_{t_{n+1}\to t_n}\int_{\mathcal{X}} P_{n+2}(\bar{t}_n,\bar{x}_n,t_{n+1},z,t_{n+2},A)\Pi_n(\bar{t}_n,\bar{x}_n,dz)\Big]$$

where

$$q_n(\bar{t}_n,\bar{x}_n) = q_n(\bar{t}_n,\bar{x}_n,\mathcal{X}-\{x_n\})$$

$$q_n(\bar{t}_n,\bar{x}_n,A) = \lim_{t_{n+1}\to t_n}\frac{P_{n+1}(\bar{t}_n,\bar{x}_n,t_{n+1},A) - \chi_A(x_n)}{t_{n+1}-t_n},$$

$$\Pi_n(\bar{t}_n,\bar{x}_n,A) = \begin{cases} \dfrac{q_n(\bar{t}_n,\bar{x}_n,A-\{x_n\})}{q_n(\bar{t}_n,\bar{x}_n)} & \text{for } q_n(\bar{t}_n,\bar{x}_n) \neq 0 \\ \chi_A(\bar{x}_n) & \text{for } q_n(\bar{t}_n,\bar{x}_n) = 0 \end{cases}$$

and q_n is a continuous function with respect to $\bar{t}_n$.

For $n = 0$ equation (2) and (3) have the form (see [3])

(5) $$\frac{\partial P_1(t_o,x_o,t_1,A)}{\partial t_1} = -\int_A q_1(t_o,x_o,t_1,z)P_1(t_o,x_o,t_1,dz) + \int_{\mathcal{X}} q_1(t_o,x_o,t_1,z)\Pi_1(t_o,x_o,t_1,z,A)P_1(t_o,x_o,t_1,dz)$$

$$\left.\frac{\partial P_2(t_o,x_o,t_1,x_o,t_2,A)}{\partial t_1}\right|_{t_1=t_o} = q_o(t_o,x_o)\Big[P_2(t_o,x_o,t_o,x_o,t_2,A) - \tag{6}$$

$$- \lim_{t_1 \to t_o} \int_{\mathcal{X}} P_2(t_o,x_ot_1,z,t_2,A)\Pi_o(t_o,x_o,dz)\Big] .$$

We shall now consider a special case, namely let the space $\mathcal{X}$ be the set of non-negative integers and let $P(X(0) = 0) = 1$. In this case we shall use the following notation

$$p_{kl}(t_1,t_2) = P(X(t_2) = l \mid X(t_1) = k),$$

$$p_{kjl}(t_o,t_1,t_2) = P(X(t_2) = l \mid X(t_1) = j,\ X(t_o) = k)$$

$$q_{kjl}(t_o,t_1) = \lim_{t_2 \to t_1} \frac{p_{kjl}(t_o,t_1,t_2)}{t_2 - t_1} \qquad \text{for } j \neq l$$

$$q_{kj}(t_o,t_1) = \lim_{t_2 \to t_1} \frac{1 - p_{kjj}(t_o,t_1,t_2)}{t_2 - t_1}$$

$$q_{kl}(t_1) = \lim_{t_2 \to t_1} \frac{p_{kl}(t_1,t_2)}{t_2 - t_1} \qquad \text{for } k \neq l$$

$$q_k(t_1) = \lim_{t_2 \to t_1} \frac{1 - p_{kk}(t_1,t_2)}{t_2 - t_1} .$$

Let

$$q_{kjl} \begin{cases} = 0 & \text{for } l \neq j,j+1 \\ \neq 0 & \text{for } l = j,j+1 \end{cases} \qquad q_{kl} \begin{cases} = 0 & \text{for } l \neq k,k+1 \\ \neq 0 & \text{for } l = k,k+1 \end{cases}$$

This implies that the transition probability from the state k to a state different from k,k+1 in the interval $(t,t+\Delta t)$ is $o(\Delta t)$.

If we assume that all the above mentioned assumptions are satisfied and that the intensity functions are given, then equations (5), (6) can be written in the following form

$$\frac{\partial p_{kl}(t_1,t_2)}{\partial t_2} = - p_{kl}(t_1,t_2)q_{kl}(t_1,t_2) + \tag{7}$$
$$+ p_{kl-1}(t_1,t_2)q_{kl-1l}(t_1,t_2)$$

$$\left.\frac{\partial p_{kkl}(t_o,t_1,t_2)}{\partial t_1}\right|_{t_1=t_o} = p_{kl}(t_o,t_2)q_k(t_o) - \tag{8}$$
$$-\lim_{t_1\to t_o} p_{kk+1l}(t_o,t_1,t_2)q_{kk+1}(t_o).$$

From these equations we can find the transition probabilities as functions of the intensities.

In this paper the following problems will be considered:

A. The connection between the equations from paper [3] and the Kolmogorov equations for a Markov process will be given.

B. A theorem concerning the solutions of equations (7) and (8) will be proved.

A. The function $P_2(t_o,x_o,t_1,z,t_2,A)$ is undefined for $t_1 = t_o$ and $z \neq x_o$ moreover $P_2(t_1,z,t_1,z,t_2,A) = P_1(t_1,z,t_2,A)$.

Let us assume that there exists the limit

$$\lim_{t_o\to t_1} P_2(t_o,x_o,t_1,z,t_2,A)$$

and that for every x_o

$$\lim_{t_o\to t_1} P_2(t_o,x_o,t_1,z,t_2,A) = P_1(t_1,z,t_2,A). \tag{9}$$

It is evident that condition (9) is satisfied for a Markov process.

On the other hand condition (9) can be treated as a continuity condition.

Observe that the formula

$$\left.\frac{\partial P_2(t_o,x_o,t_1,x_o,t_2,A)}{\partial t_1}\right|_{t_1=t_o} = \frac{\partial P_1(t_o,x_o,t_2,A)}{\partial t_o} \tag{10}$$

and (9) imply that equations (5), (6) and consequently (7), (8) have the same form as the Kolmogorov equations for a Markov process.

Indeed, by assumption (9), and in view of the continuity of the function P_2 at the point t_1 and the definition of the function q_1 we can write

$$\lim_{t_0 \to t_1} q_1(t_0,x_0,t_1,z) = \lim_{t_0 \to t_1} \lim_{t_2 \to t_1} \frac{P_2(t_0,x_0,t_1,z,t_2, \mathcal{X} - \{z\})}{t_2 - t_1} =$$

$$= \lim_{t_2 \to t_1} \frac{\lim_{t_0 \to t_1} P_2(t_0,x_0,t_1,z,t_2, \mathcal{X} - \{z\})}{t_2 - t_1} =$$

$$= \lim_{t_2 \to t_1} \frac{P_1(t_1,z,t_2, \mathcal{X} - \{z\})}{t_2 - t_1} = q_0(t_1,z).$$

The function $q_1(t_0,x_0,t_1,x_1)$ is continuous at the point t_1, hence in (5) we can take

$$q_1(t_0,x_0,t_1,z) = q_0(t_1,z).$$

Moreover, by the Lebesque theorem (see [4], p. 295) we can pass in (6) to the limit under the integral sign.

From the above considerations it follows that for a completely discontinuous process conditions (9), (10) can be used in order to state that the process is a Markov process in the wide sense (see [1], pp. 240-247).

B. Let the space $\mathcal{X}$ be the set of non-negative integers. In this case condition (2) can be written in the following form

$$(11) \qquad p_{kl}(t_1,t_1) = \delta_k(l) = \begin{cases} 1 & \text{for } l = k \\ 0 & \text{for } l \neq k \end{cases}$$

and condition (9) in the form

$$(12) \qquad \lim_{t_0 \to t_1} p_{kjl}(t_0,t_1,t_2) = p_{jl}(t_1,t_2).$$

It follows from (12) that

$$q_{kjl}(t_o,t_1) = q_{jl}(t_1) \qquad \text{for} \quad j \neq l$$

$$q_{kj}(t_o,t_1) = q_j(t_1)$$

Now we are going to prove the following theorem.

Theorem. If X(t) is a completely discontinuous stochastic process, and if conditions (11) and (12) hold as well as the intensity functions are given by the formulas

$$(13) \qquad q_{kl}(t) = \begin{cases} f'(t) & \text{for} \quad l=k,k+1 \\ 0 & \text{for} \quad l\neq k,k+1 \end{cases}$$

where f(t) is a increasing function of class C^1 and $f(0) > 0$, then the unique solution of equations (7), (8) is

$$(14) \qquad p_{kl}(t_1,t_2) = \frac{\left[f(t_2) - f(t_1)\right]^{l-k}}{(l-k)!} e^{-[f(t_2)-f(t_1)]}$$

and X(t) is a Markov process in the wide sense.

Proof. Let the assumptions of the theorem hold.Equation (7) can be written in the form

$$(15) \qquad \begin{aligned} \frac{\partial p_{kl}(t_1,t_2)}{\partial t_2} &= -p_{kl}(t_1,t_2)f'(t_2) + \\ &+ p_{kl-1}(t_1,t_2)f'(t_2), \quad l>k \end{aligned}$$

moreover

$$p_{kk-1}(t_1,t_2) = 0, \quad p_{kl}(t_1,t_2) = \delta_k(l).$$

Let k and the parameter t_1 be fixed. Then (15) is a recurrent system of linear differential equations. The theorems concernning the existence and the uniqueness of the solution of such systems are well known. It is easy to verify that if

(14) is the solution of system (15), then (14) is the unique solution of this system.

If we assume that all the assumptions mentioned above hold then it is also easy to see that transition probabilities (14) satisfy equation (8).

Thus it follows from the general theory of differential equations that (14) is the unique solution of (7) and (8). It is evident that if transition probabilities (14) satisfy the Chapmann-Kolmogorov equation then the considered process is a Markov process in the wide sense. The proof of the theorem is complete.

It is interesting that transition probabilities (14) have the same form as the corresponding transition probabilities in the Poisson process but they where derived without the assumption that the process is a process with independent increments.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW

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